1. Sup

A set of real numbers $E$ is called *bounded from above* if there exists a number $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in E$.

A number $M \in \mathbb{R}$ is called the *supremum* or also the *least upper bound* of $E$ if

- $M$ is an upper bound for $E$, and:
- any other upper bound $M'$ of $E$ satisfies $M' \geq M$.

**Exercise 1.** Show that if $M_1$ and $M_2$ are both least upper bounds of $E$ then $M_1 = M_2$; in other words, there is at most one least upper bound.

**Exercise 2.** Let $E \subset \mathbb{R}$ be a nonempty bounded set. Prove that if $M$ is an upper bound for $E$ then so is any other number $M' \geq M$.

For any set $E \subset \mathbb{R}$ we write

$$\sup E$$

for the least upper bound of $E$, if $E$ has one.

To prove that a given number $M$ is the supremum of a given set $E \subset \mathbb{R}$ you must show (1) that $M$ is an upper bound for $E$, i.e. every $x \in M$ satisfies $x \leq M$ and (2) that there is no smaller upper bound, i.e. if $M' < M$ then $E$ contains a number $x$ with $x > M'$.

**Exercise 3.** Find the least upper bounds of the sets

1. $E_1 = \{x \in \mathbb{R} \mid x \leq 2\}$
2. $E_2 = \{x \in \mathbb{R} \mid x < 2\}$
3. $E_3 = \{x \in \mathbb{R} \mid x = \frac{n}{n+1} \text{ for some } n \in \mathbb{N}\}$

**Exercise 4.** Let $A, B \subset \mathbb{R}$ be two given nonempty sets, and suppose $a = \sup A$, $b = \sup B$. Are the following statements True or False? (i.e. provide a proof or a counterexample for each statement.)

1. $a \in A$,
2. $\sup(A \cup B) = \max\{a, b\}$
3. $\sup(A \cap B) = \min\{a, b\}$
4. if $A \subset B$ and $A \neq B$ then $a < b$.

2. The least upper bound axiom

The following is a basic property of the real numbers:

*Any nonempty set $E \subset \mathbb{R}$ which has an upper bound also has a least upper bound.*

Depending on how you define the real numbers this is either an Axiom (as in Apostol's volume 1) or a theorem which must be proved from other axioms concerning the real numbers.

This semester we shall use the “sup” in the definition of the (Riemann integral).
Exercise 5. The proof that a number exists whose square equals 2.
Consider the set \( E \equiv \{ x \in \mathbb{R} \mid x \geq 0 \text{ and } x^2 < 2 \} \).

1. Show that \( E \) is nonempty and bounded from above.
2. Let \( \alpha = \sup E \). Prove that \( \alpha^2 = 2 \).

You may use the following fact, whose proof is "elementary" but messy: if \( x > 0 \) satisfies \( x^2 < 2 \) then there exists a real number \( y > x \) such that \( y^2 < 2 \); if \( x > 0 \) satisfies \( x^2 > 2 \) then there exists a real number \( 0 < y < x \) with \( y^2 > 2 \).

3. Inf

Instead of upper bounds one can also consider lower bounds for sets \( E \subset \mathbb{R} \). You should write the definitions for

- lower bound of \( E \)
- greatest lower bound, or infimum of \( E \)

and formulate the greatest lower bound axiom.

Exercise 6. Show that the greatest lower bound property follows from the least upper bound property.

4. Solutions

Exercise 1. Since \( M_1 \) is the least upper bound for \( E \) and since \( M_2 \) is an upper bound it follows that \( M_1 \leq M_2 \).

Since \( M_2 \) is the least upper bound for \( E \) and since \( M_1 \) is an upper bound it follows that \( M_2 \leq M_1 \).

So we have \( M_1 \leq M_2 \) and \( M_2 \leq M_1 \), which implies \( M_1 = M_2 \).

Exercise 2. To prove that \( M' \) is an upperbound for \( E \) you must show that \( x \leq M' \) for all \( x \in E \).

But \( M \) is an upper bound, so we know \( x \leq M \) for all \( x \in E \).

Since \( M < M' \) we also have \( x \leq M' \) for all \( x \in E \).

Therefore \( M' \) is indeed an upper bound for \( E \).

Exercise 3.

1. \( M = 2 \) is an upper bound since, by definition, every \( x \in E \) satisfies \( x \leq 2 \).

If \( M' \) is an upperbound for \( E \), then \( 2 \in E \) implies \( 2 \leq M' \). Hence there can be no smaller upper bound than \( M = 2 \).

Thus \( \sup E = 2 \).

2. Again \( M = 2 \) is an upper bound since, by definition, every \( x \in E \) satisfies \( x < 2 \) and hence \( x \leq 2 \).

In this problem \( 2 \notin E \) so it is conceivable that there is a smaller upper bound for \( E \) than \( M = 2 \). We now show this is actually not the case:

Consider any number \( M' < 2 \). Then define \( x = (M' + 2)/2 \).

Since \( M' < 2 \) we have \( x < (2+2)/2 = 2 \), so that \( x \in E \).

Since \( M' < 2 \) we also have \( x > (M' + M')/2 = M' \).

Hence we have a number \( x \in E \) with \( x > M' \). Therefore \( M' \) cannot be an upper bound for \( E \) if \( M' < 2 \).

Conclusion: \( \sup E = 2 \).

3. \( \sup E = 1 \).

Exercise 4.

1. False. See exercise 3(2).

2. True.
(3). False. In the easiest counterexample $A = \{0\}$, $B = \{1\}$ and $A \cap B$ is empty so that $\sup A \cap B$ does not exist.

Another example: $A = \{0,1\}$ and $B = \{0,2\}$, so that $a = 1$, $b = 2$, but $A \cap B = \{0\}$ so $\sup A \cap B = 0 < \min \{a,b\}$.

(4). False. Consider $A = E_1$ and $B = E_2$ from exercise 3.

Exercise 5. See volume I of Apostol’s CALCULUS.