A particle’s path is described by the parametric equation \( \mathbf{r}(t) = \sin(t^2) \mathbf{i} + \cos^2 t \mathbf{j} \), for \( t > 0 \).

a. (6 points) Compute \( \mathbf{r}(0) \cdot \mathbf{r}(\pi/4) \) and \( \mathbf{r}(0) \times \mathbf{r}(\pi/4) \).

Solution:
\( \mathbf{r}(0) = \sin(0) \mathbf{i} + \cos^2(0) \mathbf{j} = \mathbf{j} \) and
\( \mathbf{r}(\pi/4) = \sin(\pi^2/16) \mathbf{i} + \cos^2(\pi/4) \mathbf{j} = \sin(\pi^2/16) \mathbf{i} + \frac{1}{2} \mathbf{j} \)
So \( \mathbf{r}(0) \cdot \mathbf{r}(\pi/4) = (0)(\sin(\pi^2/16)) + (1)(1/2) = 1/2 \) and
\[
\mathbf{r}(0) \times \mathbf{r}(\pi/4) = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ \sin(\pi^2/16) & \frac{1}{2} & 0 \end{array} \right|
= \left[ (1)(0) - \left( \frac{1}{2} \right)(0) \right] \mathbf{i} - \left[ (0)(0) - \left( \sin(\pi^2/16) \right)(0) \right] \mathbf{j} + \left[ (0)(\frac{1}{2}) - (1) \left( \sin(\pi^2/16) \right) \right] \mathbf{k}
= -\sin(\pi^2/16) \mathbf{k}

b. (4 points) Find \( \mathbf{T}(t) \), the unit tangent vector for the particle’s motion.

Solution:
We have that \( \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \).
Now \( \mathbf{r}'(t) = 2t \cos(t^2) \mathbf{i} - 2 \sin t \cos t \mathbf{j} \). This gives us that \( |\mathbf{r}'(t)| = \sqrt{(2t \cos(t^2))^2 + (-2 \sin t \cos t)^2} = \sqrt{4t^2 \cos^2(t^2) + 4 \sin^2 t \cos^2 t} = 2 \sqrt{t^2 \cos^2(t^2) + \sin^2 t \cos^2 t} \).
Putting this all together, we have that
\[
\mathbf{T}(t) = \frac{2t \cos(t^2)}{2 \sqrt{t^2 \cos^2(t^2) + \sin^2 t \cos^2 t}} \mathbf{i} - \frac{2 \sin t \cos t}{2 \sqrt{t^2 \cos^2(t^2) + \sin^2 t \cos^2 t}} \mathbf{j}
\]
\[
= \frac{t \cos(t^2)}{\sqrt{t^2 \cos^2(t^2) + \sin^2 t \cos^2 t}} \mathbf{i} - \frac{\sin t \cos t}{\sqrt{t^2 \cos^2(t^2) + \sin^2 t \cos^2 t}} \mathbf{j}
\]