Find the flux of the vector field \( \mathbf{F} = xy\mathbf{i} + 2xz\mathbf{j} - zk \) through the portion of the plane \( x + y + z = 1 \) in the first octant in the direction pointing away from the origin.

Solution: We first note that, since \( x + y + z = 1 \), we have that \( z = 1 - x - y \) and so the plane is the graph of the function \( f(x, y) = 1 - x - y \). We also have that the "shadow" of the surface in the \( xy \) plane is the triangle bounded by \( x = 0, y = 0 \) and \( x + y = 1 \).

Now the flux is equal to \( \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma \) where \( S \) represents the plane and \( \mathbf{n} \) is an appropriately chosen normal vector. In particular, the normal to the surface that points away from the origin always points up in this particular case (we can use the picture to determine this), so we want to choose \( \mathbf{n}_{up} \). Therefore, the flux is equal to

\[
\iint_S \mathbf{F} \cdot \mathbf{n}_{up} \, d\sigma = \int_{y=0}^{1} \int_{x=0}^{1-y} \left( xy, 2x, -(1-x-y) \right) \cdot \left( -f_x, -f_y, 1 \right) \, dx \, dy
\]

\[
= \int_{y=0}^{1} \int_{x=0}^{1-y} \left( xy, 2x, -(1-x-y) \right) \cdot (1, 1, 1) \, dx \, dy
\]

\[
= \int_{y=0}^{1} \int_{x=0}^{1-y} xy + 3x + y - 1 \, dx \, dy
\]

Note 1: Don’t forget to replace \( z \) with \( f(x, y) \)!

Note 2: We could just as easily integrate in the order \( dy \, dx \).

Note 3: Our limits of integration come from the fact that we are integrating over the "shadow" of the surface in the \( xy \) plane.

It remains now to carry out this integration

\[
\iint_S \mathbf{F} \cdot \mathbf{n}_{up} \, d\sigma = \int_{y=0}^{1} \int_{x=0}^{1-y} xy + 3x + y - 1 \, dx \, dy = \int_{y=0}^{1} \left[ \frac{1}{2} x^2 y + \frac{3}{2} x^2 + xy - x \right]_{x=0}^{x=1-y} \, dy
\]

\[
= \int_{y=0}^{1} \left[ \frac{1}{2} (1-y)^2 y + \frac{3}{2} (1-y)^2 + (1-y)^2 - (1-y) \right] \, dy
\]

\[
= \int_{y=0}^{1} \left[ \frac{1}{2} y^3 - \frac{1}{2} y^2 - \frac{1}{2} y + \frac{1}{2} \right] \, dy = \left[ \frac{1}{2} y - \frac{1}{4} y^2 - \frac{1}{6} y^3 + \frac{1}{8} y^4 \right]_{y=0}^{y=1}
\]

\[
= \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \frac{1}{8} = \frac{5}{24}
\]