Let \( \mathbf{F} = yi + (x + 3y^2z)j + (y^3 + \cos z)k \)

1. (6 points) Find a potential function for \( \mathbf{F} \).

Solution: We want to find a function \( f(x, y, z) \) such that \( \nabla f = \mathbf{F} \). That is, we need to find \( f \) such that

\[
\begin{align*}
  f_x &= y \quad (1) \\
  f_y &= x + 3y^2z \quad (2) \\
  f_z &= y^3 + \cos z \quad (3)
\end{align*}
\]

Integrating Equation (1) with respect to \( x \) yields

\[ f(x, y, z) = \int y \, dx = xy + g(y, z) \quad (4) \]

Here \( g(y, z) \) is an arbitrary function of \( y \) and \( z \). Taking the partial derivative of Equation (4) with respect to \( y \) gives us \( f_y = x + g_y \). That, along with Equation (2), gives us that \( x + 3y^2z = x + g_y \) and therefore \( g_y = 3y^2z \). Integrating that with respect to \( y \) gives us that \( g(y, z) = \int 3y^2z \, dy = y^3z + h(z) \), where \( h(z) \) is an arbitrary function of \( z \). Substituting this expression into Equation (4) yields

\[ f(x, y, z) = xy + y^3z + h(z) \quad (5) \]

Taking the partial derivative of Equation (5) with respect to \( z \) gives us that \( f_z = y^3 + h'(z) \). That, along with Equation (2), gives us that \( y^3 + \cos z = y^3 + h'(z) \). So \( h'(z) = \cos z \) and therefore \( h(z) = \sin z + C \) where \( C \) is an arbitrary constant. Therefore we have that

\[ f(x, y, z) = xy + y^3z + \sin z + C \]

is a potential function for \( \mathbf{F} \) for any choice of the constant \( C \).

2. (4 points) Do we have enough information to find \( \int_{(0,0,0)}^{(3,2,\pi/2)} \mathbf{F} \cdot d\mathbf{r} \) for this \( \mathbf{F} \)?

Explain why or why not. If we do have enough information, evaluate \( \int_{(0,0,0)}^{(3,2,\pi/2)} \mathbf{F} \cdot d\mathbf{r} \).

Solution: The existence of the potential function \( f \) tells us that \( \mathbf{F} \) is a conservative vector field. Therefore \( \int_{(0,0,0)}^{(3,2,\pi/2)} \mathbf{F} \cdot d\mathbf{r} \) is independent of the path taken from \((0, 0, 0)\) to \((3, 2, \pi/2)\) and therefore we have enough information to evaluate the integral. Using the Fundamental Theorem for line integrals then gives us that

\[ \int_{(0,0,0)}^{(3,2,\pi/2)} \mathbf{F} \cdot d\mathbf{r} = f(3, 2, \pi/2) - f(0, 0, 0) \]

where \( f \) is the potential function that we found in part (a). So

\[ \int_{(0,0,0)}^{(3,2,\pi/2)} \mathbf{F} \cdot d\mathbf{r} = 7 + 4\pi \]