1a. (8 points) Find the work done by the force field $F(x, y) = xy\mathbf{i} + 2y^2\mathbf{j}$ in moving a particle along the quarter-circle $r(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ for $\frac{\pi}{2} \leq t \leq \pi$.

Solution: The parametrization gives us that $x(t) = \cos t$, $y(t) = \sin t$, $x'(t) = -\sin t$ and $y'(t) = \cos t$. Then the work is given by

$$W = \int_C F \cdot dr = \int_{\pi/2}^{\pi} (\cos t \sin t \mathbf{i} + 2\sin^2 t \mathbf{j}) \cdot (-\sin t \mathbf{i} + \cos t \mathbf{j}) \, dt$$

$$= \int_{\pi/2}^{\pi} (-\sin^2 t \cos t + 2\sin^2 t \cos t) \, dt = \int_{\pi/2}^{\pi} \sin^2 t \cos t \, dt$$

$$= \int_{u=1}^{0} u^2 \, du = \left[ \frac{1}{3} u^3 \right]_{u=1}^{0} = -\frac{1}{3}$$

In the third line we used the substitution $u = \sin t$.

b. (1 point) What does it mean for a vector field to be conservative?

Hint: The answer has nothing to do with how the vector field plans to vote next year.

Solution: There are several equivalent conditions for a vector field to be conservative, and therefore several possible answers. One possible answer is that a vector field $F$ is conservative if there exists a potential function $f$, that is, if there exists $f(x, y, z)$ such that $\nabla f = F$. Another possible answer is that the work done by a conservative vector field is path-independent. Yet another possible answer is that $F$ is conservative if the work done by $F$ over any closed curve $C$ is zero.

c. (1 point) Is $F$ (from part a) a conservative vector field?

Solution: No. We can see this by applying the component test. If we write $F = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, then $M_y = x$, but $N_x = 0 \neq x$. Since $M_y \neq N_x$, we have that $F$ is not conservative.