1. (2 points) State the Divergence Theorem. (Include all supposition needed to use the theorem)

Solution: If the surface \( S \) is the boundary of the region \( D \) then
\[
\iint_S \mathbf{F} \cdot \mathbf{n}_{\text{out}} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dx \, dy \, dz
\]
where \( \mathbf{n}_{\text{out}} \) is the outward unit normal to \( S \).

2. (8 points) Use the Divergence Theorem to calculate the flux of the vector field \( \mathbf{F} = (x + \sin y)i + (xy + xz)j + (y^2 - 2x)k \) outward across the surface \( S \), where \( S \) is the boundary of the region \( D \) bounded above by the plane \( y + z = 1 \) and below by the plane \( z = 0 \) with \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \).

Solution: Since \( S \) is the boundary of \( D \) we can apply the Divergence Theorem. To use the theorem, we need to calculate the divergence of \( \mathbf{F} \).
\[
\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (x + \sin y) + \frac{\partial}{\partial y} (xy + xz) + \frac{\partial}{\partial z} (y^2 - 2x) = 1 + x
\]
So we have that \( \iint_S \mathbf{F} \cdot \mathbf{n}_{\text{out}} \, d\sigma = \iiint_D (x + 1) \, dx \, dy \, dz \).

If we integrate in the order \( dz \, dy \, dx \), then our integral will become
\[
\int_{x=0}^1 \int_{y=0}^1 \int_{z=0}^{1-y} (x + 1) \, dz \, dy \, dx
\]
We can obtain similar expressions if we integrate in other orders such as \( dz \, dx \, dy \). To find the flux, we then compute this integral.

\[
\iint_S \mathbf{F} \cdot \mathbf{n}_{\text{out}} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dy \, dx = \int_{x=0}^1 \int_{y=0}^1 \int_{z=0}^{1-y} (x + 1) \, dz \, dy \, dx
\]
\[
= \int_{x=0}^1 \int_{y=0}^1 [\,(x + 1)z]_{z=0}^{1-y} \, dy \, dx = \int_{x=0}^1 \int_{y=0}^1 (x + 1)(1 - y) \, dy \, dx
\]
\[
= \int_{x=0}^1 \int_{y=0}^1 (-xy - y + x + 1) \, dy \, dx
\]
\[
= \int_{x=0}^1 \left[ -\frac{1}{2}xy^2 - \frac{1}{2}y^2 + xy + y \right]_{y=0}^1 \, dx = \int_{x=0}^1 \left( \frac{1}{2}x + \frac{1}{2} \right) \, dx
\]
\[
= \left[ \frac{1}{4}x^2 + \frac{1}{2}x \right]_{x=0}^1 = 3/4
\]