Math 234 Practice Problems for Exam 1

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Directions: Answer each of the following questions. Solutions are on the last page.

Problem 1. Find each of the following limits, or show that they don’t exist.

(a) \( \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} \)

(b) \( \lim_{(x,y) \to (0,0)} \frac{xy^2}{(x^2 + y^2) \cos(x^2 + y^2)} \)

(c) \( \lim_{(x,y) \to (0,1)} \frac{x + y - 1}{xy} \)

Problem 2. Let \( z = f(x, y) = \frac{1}{2}x^2y + \sin y \). Suppose that \( x = x(t) \) and \( y = y(t) \) are differentiable functions of \( t \). Suppose further that we know the following:

(i) When \( t = 0 \) we have that \( x = 2 \) and \( y = \pi \)

(ii) \( \frac{dx}{dt} \bigg|_{t=0} = \frac{dy}{dt} \bigg|_{t=0} \)

(iii) \( \frac{dz}{dt} \bigg|_{t=0} = e \)

Find the derivatives \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \) at \( t = 0 \).
Problem 3. Find all directions \( \mathbf{u} \) such that the directional derivative of \( f(x, y) = x^2 + y \) at the point \( P(2, 5) \) is equal to 1.

Problem 4. Suppose the equation \( xz + 3x^2y - e^{yz} = 0 \) defines \( z \) as a differentiable function of \( x \) and \( y \). Let \( P_0 = (1, 0, 1) \).

(a) Find the direction \( \mathbf{u} \) in which \( z \) increases most rapidly at \( P_0 \). What is the derivative of \( z \) in that direction?

(b) Find the direction \( \mathbf{u} \) in which \( z \) decreases most rapidly at \( P_0 \). What is the derivative of \( z \) in that direction?

(c) Find all directions \( \mathbf{u} \) in which the directional derivative of \( z \) at \( P_0 \) in direction \( \mathbf{u} \) is equal to zero.

Problem 5. Suppose that \( f(x, y) \) is a differentiable function such that the directional derivative of \( f \) at \( P_0(1, 2) \) in the direction pointing towards \( P_1(3, 4) \) is \( \sqrt{2} \) and the directional derivative of \( f \) at \( P_0 \) in the direction pointing towards \( P_2(2, 0) \) is \( \sqrt{5} \).

(a) Is there a direction \( \mathbf{u} \) in which the rate of change of \( f \) in direction \( \mathbf{u} \) at \( P_0 \) is equal to 4?

(b) Is there a direction \( \mathbf{v} \) in which the rate of change of \( f \) in direction \( \mathbf{v} \) at \( P_0 \) is equal to \(-4\)?
Solutions

1a. Does not exist.

1b. 0

1c. Does not exist.

2. \[ \frac{dx}{dt} \bigg|_{t=0} = \frac{dy}{dt} \bigg|_{t=0} = \frac{e}{2\pi + 1} \]

3. \( j \) and \( \frac{8}{17} i - \frac{15}{17} j \)

4a. \( u = -\frac{1}{\sqrt{5}} i - \frac{2}{\sqrt{5}} j \). \((Du f)_{P_0} = \sqrt{5}\)

4b. \( u = \frac{1}{\sqrt{5}} i + \frac{2}{\sqrt{5}} j \). \((Du f)_{P_0} = -\sqrt{5}\)

4c. \(-\frac{1}{\sqrt{5}} i + \frac{2}{\sqrt{5}} j \) and \(\frac{1}{\sqrt{5}} i - \frac{2}{\sqrt{5}} j\)

5a. No. The rate of fastest increase of \( f \) at \( P_0 \) is \( \sqrt{10} \), which is less than 4.

5b. No. The rate of fastest decrease of \( f \) at \( P_0 \) is \( -\sqrt{10} \), which is greater than \(-4 \).