Math 234 Review Problems for Exam 2

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Directions: Answer each of the following questions. Solutions are on the pages 3 and 4.

Problem 1. Find the volume of the solid in the first octant bounded by the coordinate planes and the plane passing through the points $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 1)$.

Problem 2a. Give an example of a (non-constant) vector field $\mathbf{F} = M(x, y, z)\,\mathbf{i} + N(x, y, z)\,\mathbf{j} + P(x, y, z)\,\mathbf{k}$ which is conservative. Justify why the field is conservative.

Problem 2b. Give an example of a vector field $\mathbf{F} = M(x, y, z)\,\mathbf{i} + N(x, y, z)\,\mathbf{j} + P(x, y, z)\,\mathbf{k}$ which is not conservative. Justify why the field is not conservative.

Note: Try to come with with your vector fields without looking at any examples in the book. The idea is that if you are asked to give an example of a vector field that is (or isn't) conservative, you should know how to come up with one on the spot.

Problem 3a. State Green's Theorem (both forms).

Problem 3b. Let $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ be a conservative vector field and suppose C is a curve for which Green's Theorem applies. What happens when we apply Green's Theorem to find the counterclockwise circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$?

Hint: For problems 4 and 5, try to find a relatively fast way to come up with the answer (i.e. one that doesn't require you to parametrize a bunch of curves). For Problem 4 drawing a picture of the curve may be helpful.

Problem 4. Let $\mathbf{F} = \left(\cos(x^5) - \frac{1}{3}y^3\right)\mathbf{i} + \frac{1}{3}x^3\,\mathbf{j}$. Find the counterclockwise circulation $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve consisting of the line segment joining $(-2,0)$ and $(-1,0)$, the half circle $y = \sqrt{1-x^2}$, the line segment joining $(1,0)$ and $(2,0)$ and the half circle $y = \sqrt{4-x^2}$.

Problem 5. Let $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k}$. Find the work $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve consisting of the line segment joining $(1,0,0)$ and $(0,0,0)$, followed by the parabola $z = x^2$, $y = 0$ joining $(0,0,0)$ and $(1,0,1)$, followed by the line segment joining $(1,0,1)$ and $(1,1,1)$, followed by the line segment joining $(1,1,1)$ and $(1,1,2)$.

Solutions

1. $\frac{11}{6}$.

2a. Many solutions are possible, but one example would be the following:

The vector field $\mathbf{F} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$ is conservative because it is the gradient of the potential function $f(x, y, z) = x^2 + y^2 + z^2$.
Note that a good way to come up with a conservative vector field is to start with a function of three variables (or two vatiables if we want a 2 dimensional vector field) and then to calculate its gradient, which will be a conservative vector field.

2b. Again, many solutions are possible. One example would be the following:

The vector field $\mathbf{F} = y\,\mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$ is not conservative because it fails the component test. $\frac{\partial M}{\partial y} = 1$ but $\frac{\partial N}{\partial x} = 0 \neq 1$.
Note that a good way to come up with a vector field that is not conservative is to choose M, N and P that will fail the component test.

3a. (Tangential Form) If C is a "nice" simple closed curve that encloses a region R and $\mathbf{F} = M(x,y)\mathbf{i} + N(x.y)\mathbf{j}$ is a "nice" vector field, then the counterclockwise circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is equal to the double integral

$$\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dx\, dy$$

(Normal Form) Under the same hypotheses as above, the outward flux across C, $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C M\, dy - N\, dx$ is equal to the double integral

$$\iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\, dx\, dy$$

3b. Since $\mathbf{F}$ is conservative we have that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Therefore applying Green's Theorem tells us that the counterclockwise circulation will be equal to $\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dx\, dy = \iint_R 0\, dx\, dy = 0$. This result shouldn't surprise us because $\mathbf{F}$ is a conservative vector field and therefore its work done around any closed curve (which is equal to the circulation) is always zero.

4. The answer is $\frac{15\pi}{4}$. The best way to do this problem is to use Green's Theorem to turn the problem into a double integral and then to change to polar coordinates to evaluate the double integral.

5. The answer is 5. The trick here is that $\mathbf{F}$ is a conservative vector field and therfore the integral does not depend on the path we take from $(1,0,0)$ to $(1,1,2)$. Once we know that we can find the work by first finding a potential function and then evaluating that potential at the given points.