Finding Potential Functions

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1 Introduction

Given a vector field $\mathbf{F}$, one thing we may be asked is to find a potential function for $\mathbf{F}$. That is, we want to find a scalar-valued function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$. In general, we cannot guarantee the exisance of such a function. To establish the exisance of a potential for $\mathbf{F}$ (that is, to show that $\mathbf{F}$ is conservative), we can use the Component Test given on Page 1164 of the text. If it is the case that $\mathbf{F}$ is conservative, then we can find the potential $f$ through a systematic procedure that is best illustrated by example.

2 An Example

2.1 Setting up the problem

Let $\mathbf{F}$ be the vector field $2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2z)\mathbf{k}$. Find a potential function for $\mathbf{F}$.

One can use the component test to show that $\mathbf{F}$ is conservative, but we will skip that step and go directly to finding the potential. We want to find $f$ such that $\nabla f = \mathbf{F}$. That is we want to have

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2z)\mathbf{k}$$

So the problem is to find a function $f$ such that...
\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2xy \\
\frac{\partial f}{\partial y} &= x^2 + 2yz \\
\frac{\partial f}{\partial z} &= y^2 + 2z
\end{align*}
\]

### 2.2 Step 1: Finding a preliminary form for \( f \)

We have that \( \frac{\partial f}{\partial x} = 2xy \). Therefore \( f \) is given by the indefinite integral \( f(x, y, z) = \int 2xy \, dx \). Solving this yields

\[
f(x, y, z) = \int 2xy \, dx = x^2y + C(y, z)
\]

Note that the book uses the notation \( g(y, z) \) instead of \( C(y, z) \). Now, before proceeding any further, we explore why the term \( C(y, z) \) appears in our expression, and in particular why we must allow it to depend on \( y \) and \( z \).

### 2.3 The reason for the term \( C(y, z) \)

When we calculate a single variable indefinite integral, we need a constant of integration \( C \). For example \( \int x \, dx = \frac{1}{2}x^2 + C \). The reason for the \( C \) term is that we want the most general antiderivative we can find. The function \( f_1(x) = \frac{1}{2}x^2 \) has the property that \( f_1'(x) = x \), but so do the functions \( f_2(x) = \frac{1}{2}x^2 + 5 \), \( f_3(x) = \frac{1}{2}x^2 - \pi \) and, in general \( f(x) = \frac{1}{2}x^2 + C \) for any constant \( C \).

Similarly, in our case we want (for the time being) to find the most general function \( f(x, y, z) \) that has the property that \( \frac{\partial f}{\partial x} = 2xy \). The function \( f_1(x, y, z) = x^2y \) has the property that \( \frac{\partial f_1}{\partial x} = 2xy \), but so each of the following functions

\[
\begin{align*}
f_2(x, y, z) &= x^2y - 2 \\
f_3(x, y, z) &= x^2y + yz \\
f_4(x, y, z) &= x^2y + \sin ye^{2z-\cos(yz)} + \tan^{-1}\left(\frac{yz}{y^2 + z^2 + 1}\right)
\end{align*}
\]
In general \( f(x, y, z) = x^2y + C(y, z) \) will have the desired property for any function \( C(y, z) \). Hence, the most general form of \( f \) with the property that \( \frac{\partial f}{\partial x} = 2xy \) is \( f(x, y, z) = x^2y + C(y, z) \) and it is therefore the indefinite integral.

We now return to the problem of finding the potential.

### 2.4 Step 2: Refining \( f \)

We have found an \( f \) (actually an infinite family of \( f \)'s) such that \( \frac{\partial f}{\partial x} = 2xy \).

But we also have conditions on \( \frac{\partial f}{\partial y} \) and \( \frac{\partial f}{\partial z} \). We first work with our condition for \( \frac{\partial f}{\partial y} \). On the one hand, by Equation (2) we have that

\[
\frac{\partial f}{\partial y} = x^2 + 2yz
\]

On the other hand, taking the partial derivative of both sides of Equation (4) with respect to \( y \) gives us that

\[
\frac{\partial f}{\partial y} = x^2 + \frac{\partial C}{\partial y}
\]

(5)

Taken together, Equations (2) and (5) imply that \( x^2 + 2yz = x^2 + \frac{\partial C}{\partial y} \) and therefore

\[
\frac{\partial C}{\partial y} = 2yz
\]

Integrating this result gives us that

\[
C(y, z) = \int 2yz \, dy = y^2z + C_1(z)
\]

(6)

The reason for having \( C_1(z) \) in Equation (6) is the same as the reason for having \( C(y, z) \) in Equation (4). That is, \( C(y, z) = y^2z + C_1(z) \) has the property that \( \frac{\partial C}{\partial y} = 2yz \) for any choice of \( C_1(z) \) and (at the moment) we want the most general expression for \( C(y, z) \) that we can find.
Now, substituting our expression for $C(y, z)$ from Equation (6) into Equation (4) gives us that

$$f(x, y, z) = x^2 y + y^2 z + C_1(z)$$  (7)

2.5 Step 3: Obtaining the final answer

The expression we have obtaining for $f$ in Equation (7) has the property that

$$\frac{\partial f}{\partial x} = 2xy$$

and

$$\frac{\partial f}{\partial y} = x^2 + 2yz.$$  

Now we use our condition on $\frac{\partial f}{\partial z}$, proceeding in a similar manner as in Step 2. On the one hand, Equation (3) tells us that

$$\frac{\partial f}{\partial z} = y^2 + 2z$$

On the other hand, taking the partial derivative of both sides of Equation (7) gives us that

$$\frac{\partial f}{\partial z} = y^2 + \frac{dC_1}{dz}$$  (8)

Taken together, Equations (3) and (8) imply that $y^2 + 2z = y^2 + \frac{dC_1}{dz}$ and therefore

$$\frac{dC_1}{dz} = 2z$$

Integrating the result gives us that

$$C_1(z) = \int 2z \, dz = z^2 + C_2$$  (9)

Where $C_2$ is an arbitrary constant. Substituting our expression for $C_1(z)$ from Equation (9) into Equation (7) gives us that

$$f(x, y, z) = x^2 y + y^2 z + z^2 + C_2$$  (10)

We can now verify by direct calculation that

$$\nabla f = 2xy \mathbf{i} + (x^2 + 2yz) \mathbf{j} + (y^2 + 2z) \mathbf{k} = \mathbf{F}$$

Therefore $f$ as given in Equation (10) is a potential function for any choice of the constant $C_2$. That is, we can let $C_2$ be any (fixed) real number we want. In practice it is often convenient for calculations to let $C_2 = 0$. 