NORMAL FORMS FOR CR SINGULAR CODIMENSION TWO LEVI-FLAT SUBMANIFOLDS

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Abstract. Real-analytic Levi-flat codimension two CR singular submanifolds are a natural generalization to $\mathbb{C}^m$, $m > 2$, of Bishop surfaces in $\mathbb{C}^2$. Such submanifolds for example arise as zero sets of mixed-holomorphic equations with one variable antiholomorphic. We classify the codimension two Levi-flat CR singular quadrics, and we notice that new types of submanifolds arise in dimension 3 or greater. In fact, the nondegenerate submanifolds, i.e. higher order perturbations of $z_m = \bar{z}_1 z_2 + \bar{z}_2^2$, have no analogue in dimension 2. We prove that the Levi-foliation extends through the singularity in the real-analytic nondegenerate case. Furthermore, we prove that the quadric is a (convergent) normal form for a natural large class of such submanifolds, and we compute its automorphism group. In general, we find a formal normal form in $\mathbb{C}^3$ in the nondegenerate case that shows infinitely many formal invariants.

1. Introduction

Let $M \subset \mathbb{C}^{n+1}$ be a real submanifold. A fundamental question in CR geometry is to classify $M$ at a point up to local biholomorphic transformations. One approach is to find a normal form for $M$.

A real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$ is Levi-flat if the Levi-form vanishes identically. Roughly speaking, a Levi-flat submanifold is a family of complex submanifolds. Intuitively, a Levi-flat submanifold is as close to a complex submanifold as possible. In the real-analytic smooth hypersurface case, it is well-known that $M$ can locally be transformed into the real hyperplane given by

$$\text{Im } z_1 = 0. \tag{1}$$

We therefore focus on higher codimension case, in particular on codimension 2. A codimension 2 submanifold is again given by a single equation, but in this case a complex valued equation. A new phenomenon that appears in codimension 2 is that $M$ may no longer be a CR submanifold. Let $T^c_p M \subset T_p M$ be the largest subspace with $JT^c_p M = T^c_p M$, where $J$ is the complex structure on $\mathbb{C}^{n+1}$. A submanifold is CR if $\dim T^c_p M$ is constant.

Real submanifolds of dimension $n + 1$ in $\mathbb{C}^{n+1}$ with a non-degenerate complex tangent point has been studied extensively after the fundamental work of E. Bishop [1]. In $\mathbb{C}^2$, Bishop studied the submanifolds

$$w = z\bar{z} + \lambda (z^2 + \bar{z}^2) + O(3) \tag{2}$$

where $\lambda \in [0, \infty]$ is called the Bishop invariant, with $\lambda = \infty$ is interpreted as $w = z\bar{z} + O(3)$. One of Bishop’s motivations was to study the hull of holomorphy of the real submanifolds by attaching analytic discs. Bishop’s work on the family of attached analytic discs has been refined by Kenig-Webster [23,24], Huang-Krantz [19], and Huang [20]. The normal form

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theory for real submanifolds for Bishop surfaces or submanifolds was established by Moser-Webster; see \cite{10,16–18,20,28}. We would like to mention that the Moser-Webster normal form does not deal with the case of vanishing Bishop invariant.

The formal normal form and its application to holomorphic classification for surfaces with vanishing Bishop invariant was achieved by Huang-Yin \cite{20} by a completely different method. Real submanifolds with complex tangents have been studied in other situations. See for example \cite{27}, where CR singular submanifolds that are images of CR manifolds were studied. Normal forms for the quadratic part of general codimension two CR singular submanifolds in $\mathbb{C}^3$ was completely solved by Coffman \cite{9}. Huang and Yin \cite{21} studied the normal form for codimension two CR singular submanifolds of the form $w = |z_1|^2 + O(3)$. Dolbeault-Tomassini-Zaitsev \cite{12,13} and Huang-Yin \cite{22} studied CR singular submanifolds of codimension two that are boundaries of Levi-flat hypersurfaces. Burcea \cite{5} constructed the formal normal form for codimension 2 CR singular submanifolds approximating a sphere. Coffman \cite{8} found an algebraic normal form for nondegenerate CR singular manifolds in high codimension and one dimensional complex tangent.

To motivate our work, we observe that in Bishop’s work, the real submanifolds are Levi-flat away from their CR singular sets. Our purpose is to understand such submanifolds in higher dimensional case with codimension being exactly two. Notice that the latter is the smallest codimension for CR singularity to be present in (smooth) submanifolds. Regarding CR singular Levi-flat real codimension 2 submanifolds on $\mathbb{C}^{n+1}$ as a natural generalization of Bishop surfaces to $\mathbb{C}^{n+1}$, we wish to find their normal forms. For singular Levi-flat hypersurfaces and related work on foliations with singularity, see \cite{3,4,6,7,14,26}.

Our techniques revolve around the study of the Levi-map (the generalization of the Levi-form to higher codimension submanifolds) of codimension 2 submanifolds. Extending the CR structure through the singular point via Nash blowup and then extending the Levi-map to this blowup has been studied previously by Garrity \cite{15}.

A CR submanifold is \textit{Levi-flat} if the Levi-map vanishes identically. Locally, all CR real-analytic Levi-flat submanifolds of real codimension 2 can be, after holomorphic change of coordinates, written as

$$\text{Im } z_1 = 0, \quad \text{Im } z_2 = 0.$$  

(3)

If a submanifold $M$ is CR singular, denote by $M_{CR}$ the set of points where $M$ is CR. We say $M$ is Levi-flat if $M_{CR}$ is Levi-flat in the usual sense. A Levi-flat CR singular submanifold has no local biholomorphic invariants at the CR points, just as in the case of Bishop surfaces.

A real, real-analytic codimension 2 submanifold that is CR singular at the origin can be written in coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C} = \mathbb{C}^{n+1}$ as

$$w = \rho(z, \bar{z})$$

(4)

for $\rho$ that is $O(2)$. We will be concerned with submanifolds where the quadratic part in $\rho$ is nonzero in any holomorphic coordinates. We say that such submanifolds have a \textit{nondegenerate complex tangent}. For example, the Bishop surfaces in $\mathbb{C}^2$ are precisely the CR singular submanifolds with nondegenerate complex tangent.

First, let us classify the quadratic parts of CR singular Levi-flats, and in the process completely classify the CR singular Levi-flat quadrics, that is those where $\rho$ is a quadratic.

\textbf{Theorem 1.1.} Suppose that $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is a germ of a real-analytic real codimension 2 submanifold, CR singular at the origin, written in coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ as

$$w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + O(3),$$

(5)
for quadratic $A$ and $B$, where $A + B \neq 0$ (nondegenerate complex tangent). Suppose that $M$ is Levi-flat (that is $M_{CR}$ is Levi-flat).

(i) If $M$ is a quadric, then $M$ is locally biholomorphically equivalent to one and exactly one of the following:

\begin{align*}
(A.1) & \quad w = \bar{z}_1^2, \\
(A.2) & \quad w = \bar{z}_1^2 + \bar{z}_2^2, \\
& \vdots \\
(A.n) & \quad w = \bar{z}_1^2 + \bar{z}_2^2 + \cdots + \bar{z}_n^2, \\
(B.\gamma) & \quad w = |z_1|^2 + \gamma \bar{z}_1^2, \quad \gamma \geq 0, \\
(C.0) & \quad w = \bar{z}_1 \bar{z}_2, \\
(C.1) & \quad w = \bar{z}_1 \bar{z}_2 + \bar{z}_1^2.
\end{align*}

(ii) If $M$ is real-analytic, then the quadric $w = A(z, \bar{z}) + B(z, \bar{z})$ is Levi-flat, and can be put via a biholomorphic transformation into exactly one of the forms \((6)\).

By part (ii), the quadratic part in \((5)\) is an invariant of $M$ at a point. We say the type of $M$ at the origin is A.x, B.\gamma, or C.x depending on the type of the quadratic form. Following Bishop, we call types B.\gamma and A.1 Bishop-like, we could think of $\gamma = \infty$ as A.1.

By type being stable we mean that the type does not change at all complex tangents in a neighborhood of the origin under any small (or higher order) perturbations that stay within the class of Levi-flat CR singular submanifolds. As a consequence of the above theorem and because rank is lower semicontinuous, we get that the only types that are stable are A.n and C.1, although A.n are degenerate because the form $A(z, \bar{z})$ is identically zero. See also Proposition 15.1.

The quadrics $A.k$ for $k \geq 2$ do not possess a nonsingular foliation extending the Levi-foliation of $M_{CR}$ through the origin. In fact, there is a singular complex subvariety of dimension 1 through the origin contained in $M$. See §6.

In the sequel, when we wish to refer to the quadric of certain type we will use the notation $M_{C.1}$ to denote the quadric of type C.1.

The quadratic form $A(z, \bar{z})$ carries the “Levi-map” of the submanifold. Type C.1 is the unique quadric that is stable and has non-zero $A$. Having non-zero $A$ is also stable in a neighborhood of the origin under any small (or higher order) perturbations. Therefore, we say a type is non-degenerate if it is C.1 and we focus mostly on such submanifolds. First, we show that submanifolds of type C.x possess a nonsingular real-analytic foliation that extends the Levi-foliation, due to the form $A(z, \bar{z})$:

**Theorem 1.2.** Suppose that $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is a real-analytic Levi-flat CR singular submanifold of type C.1 or C.0, that is, $M$ is given by

\begin{equation}
\begin{array}{ll}
w = \bar{z}_1 z_2 + \bar{z}_1^2 + O(3) & \text{or} \\
w = \bar{z}_1 z_2 + O(3).
\end{array}
\end{equation}

Then there exists a nonsingular real-analytic foliation defined on $M$ that extends the Levi-foliation on $M_{CR}$, and consequently, there exists a CR real-analytic mapping $F: U \subset \mathbb{R}^2 \times \mathbb{C}^{n-1} \to \mathbb{C}^{n+1}$ such that $F$ is a diffeomorphism onto $F(U) = M \cap U'$, for some neighbourhood $U'$ of 0.
Here the CR structure on $\mathbb{R}^2 \times \mathbb{C}^{n-1}$ is induced from $\mathbb{C}^2 \times \mathbb{C}^{n-1}$. As a corollary of this theorem we obtain in §8 using the results of [27] that the CR singular set of any type C.1 submanifold is a Levi-flat submanifold of dimension $2n - 2$ and CR dimension $n - 2$.

The Levi-foliation on a type C.x submanifold cannot extend to a whole neighbourhood of $M$ as a nonsingular holomorphic foliation. If it did, we could flatten the foliation and $M$ would be a Cartesian product, in particular Bishop-like. Thus, the study of normal form theory for the special case when the foliation extends to a neighbourhood is reduced to the case of Bishop surfaces, which have been studied extensively.

A codimension 2 submanifold in $\mathbb{C}^m$ can arise from

$$f(z', z'') = 0$$

for a suitable holomorphic function $f$ in $m$ variables. The zero set admits two holomorphic foliations. We are interested in the case where one of foliations has leaves of maximum dimension $m - 2$, while the other has leaves of minimum dimension 0. Therefore, we will assume that $z' = z_1$ and $z'' = (z_2, \ldots, z_n)$. Functions holomorphic in some variables and anti-holomorphic in other variables, such as (8), are often called mixed-holomorphic or mixed-analytic, and come up often in complex geometry, the simplest example being the standard inner product. An interesting feature of the mixed-holomorphic setting is that the equation can be complexified into $\mathbb{C}^m$, so the sets share some of the properties of complex varieties. However, they have a different automorphism group if we wish to classify them under biholomorphic transformations. Such mixed-analytic sets are automatically real codimension 2, are Levi-flat or complex, and may have CR singularities. We study their normal form in §9. See also Theorem 1.3 below.

When a type C.1 CR singular submanifold has a defining equation that does not depend on $\bar{z}_2, \ldots, \bar{z}_n$ we prove that it is automatically Levi-flat, and it is equivalent to $M_{C.1}$.

**Theorem 1.3.** Let $M \subset \mathbb{C}^{n+1}, n \geq 2$, be a real-analytic submanifold given by

$$w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(z_1, \bar{z}_1, z_2, z_3, \ldots, z_n),$$

where $r$ is $O(3)$. Then $M$ is Levi-flat and at the origin $M$ is locally biholomorphically equivalent to the quadric $M_{C.1}$ submanifold

$$w = \bar{z}_1 z_2 + \bar{z}_1^2.$$  

The theorem is also true formally; given a formal submanifold of the form (9), it is formally equivalent to $M_{C.1}$.

A key idea in the proof of the convergence of the normalizing transformation is that the form $B(\bar{z}, z) = \bar{z}_1^2$ induces a natural mixed-holomorphic involution on quadric $M_{C.1}$. This involution also plays a key role in computing the automorphism group of the quadric in Theorem 12.4.

Finally, we also compute the automorphism group for the quadric $M_{C.1}$, see Theorem 12.4. In particular we show that the automorphism group is infinite dimensional.
Theorem 1.4. Let $M$ be a real-analytic Levi-flat type $C.1$ submanifold in $\mathbb{C}^3$. There exists a formal biholomorphic map transforming $M$ into the image of
\[
\varphi(z, \bar{z}, \xi) = (z + A(z, \xi, w)w\eta, \xi, w)
\]  
(11)
with $\eta = \bar{z} + \frac{1}{2}\xi$ and $w = \bar{z}\xi + \bar{z}^2$. Here $A = 0$, or $A$ satisfies certain normalizing conditions. When $A \neq 0$ the formal automorphism group preserving the normal form is finite or 1 dimensional.

We do not know if the formal normal form above can be achieved by convergent transformations, even if $A = 0$.

2. Invariants of codimension 2 CR singular submanifolds

Before we impose the Levi-flat condition, let us find some invariants of codimension two CR singular submanifolds in $\mathbb{C}^{n+1}$ with CR singularity at 0. Such a submanifold can locally near the origin be put into the form
\[
w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + O(3),
\]  
(12)
where $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ and $A$ and $B$ are quadratic forms. We think of $A$ and $B$ as matrices and $z$ as a column vector and write
\[
w = z^*Az + \bar{z}^tB\bar{z} + O(3),
\]  
(13)
where $\bar{z}^t = z^*$, but we write it as above and think of the term as $\bar{z}^tB\bar{z}$. The matrix $B$ is not unique. Hence we make $B$ symmetric to make the choice of the matrix $B$ canonical. The following proposition is not difficult and well-known. Since the details are important and will be used later, let us prove this fact.

Proposition 2.1. A biholomorphic transformation of [13] taking the origin to itself and preserving the form of [13] takes the matrices $(A, B)$ to
\[
(\lambda T^*AT, \lambda T^tB\bar{T}),
\]  
(14)
for $T \in GL_n(\mathbb{C})$ and $\lambda \in \mathbb{C}^\ast$. If $(F_1, \ldots, F_n, G) = (F, G)$ is the transformation then the linear part of $G$ is $\lambda^{-1}w$ and the linear part of $F$ restricted to $z$ is $Tz$.

Let us emphasize that $A$ is an arbitrary complex matrix and $B$ is a symmetric, but not necessarily Hermitian, matrix.

Proof. Let $(F_1, \ldots, F_n, G) = (F, G)$ be a change of coordinates taking
\[
w = \tilde{A}(z, \bar{z}) + \tilde{B}(\bar{z}, \bar{z}) + O(3) = \rho(z, \bar{z})
\]  
(15)
to
\[
w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + O(3).
\]  
(16)
Then
\[
G(z, \rho(z, \bar{z})) = A\left(F(z, \rho(z, \bar{z})), \bar{F}(\bar{z}, \bar{\rho}(\bar{z}, z))\right) + B\left(\bar{F}(\bar{z}, \bar{\rho}(\bar{z}, z)), \tilde{F}(\bar{z}, \bar{\rho}(\bar{z}, z))\right) + O(3)
\]  
(17)
is true for all $z$. The right hand side has no linear terms, so the linear terms in $G$ do not depend on $z$. That is, $G = \lambda^{-1}w + O(2)$, where $\lambda$ is a nonzero scalar and the negative power is for convenience.
Let $T = [T_1, T_2]$ denote the matrix representing the linear terms of $F$. Here $T_1$ is an $n \times n$ matrix and $T_2$ is $n \times 1$. Since the linear terms in $G$ do not depend on any $z_j$, $T_1$ is nonsingular. Then the quadratic terms in (17) are

$$\lambda^{-1}(\tilde{A}(z, \bar{z}) + \tilde{B}(\bar{z}, \bar{z})) = z^* T_1^* A T_1 z + \bar{z}^* T_1^* B T_1 \bar{z}. \quad (18)$$

In other words as matrices,

$$\tilde{A} = \lambda T_1^* A T_1 \quad \text{and} \quad \tilde{B} = \lambda T_1^* B T_1. \quad (19)$$

We will need to at times reduce to the 3-dimensional case, and so we need the following lemma.

**Lemma 2.2.** Let $M \subset \mathbb{C}^{n+1}$, $n \geq 3$, be a real-analytic Levi-flat CR singular submanifold of the form

$$w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + O(3), \quad (20)$$

where $A$ and $B$ are quadratic. Let $L$ be a nonsingular $(n - 2) \times n$ matrix $L$. If $A + B$ is not zero on the set $\{Lz = 0\}$, then the submanifold

$$M_L = M \cap \{Lz = 0\} \quad (21)$$

is a Levi-flat CR singular submanifold.

**Proof.** Clearly if $M_L$ is not contained in the CR singularity of $M$, then $M_L$ is a Levi-flat CR singular submanifold. $M_L$ is not contained in the CR singularity of $M$ for a dense open subset of $(n - 2) \times n$ matrices. If $M_L$ is a subset of the CR singularity of $M$, pick a CR point $p$ of $M_L$ then pick a sequence $L_n$ approaching $L$ such that $M_{L_n}$ are not contained in the CR singularity of $M$. As $A + B$ is not zero on the set $\{Lz = 0\}$, then $M_L$ is not a complex submanifold, and therefore a CR singular submanifold. Then as the Levi-form of $M_{L_n}$ vanishes at all CR points of $M_{L_n}$, the Levi-form of $M_L$ vanishes at $p$, so $M_L$ is Levi-flat. \hfill \Box

### 3. Levi-flat Quadrics

Let us first focus on Levi-flat quadrics. We will prove later that the quadratic part of a Levi-flat submanifold is Levi-flat. Let $M$ be defined in $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ by

$$w = A(z, \bar{z}) + B(\bar{z}, \bar{z}). \quad (22)$$

Being Levi-flat has several equivalent formulations. The main idea is that the $T^{(1,0)} M \times T^{(0,1)} M$ vector fields are completely integrable at CR points and we obtain a foliation of $M$ at CR points by complex submanifolds of complex dimension $n - 1$. An equivalent notion is that the Levi-map is identically zero, see [2]. The Levi-map for a CR submanifold defined by two real equations $\rho_1 = \rho_2 = 0$ (for $\rho_1$ and $\rho_2$ with linearly independent differentials) is the pair of Hermitian forms

$$i \partial \bar{\partial} \rho_1 \quad \text{and} \quad i \partial \bar{\partial} \rho_2, \quad (23)$$

applied to $T^{(1,0)} M$ vectors. The full quadratic forms $i \partial \bar{\partial} \rho_1$ and $i \partial \bar{\partial} \rho_2$ of course depend on the defining equations themselves and are therefore extrinsic information. It is important to note that for the Levi-map we restrict to $T^{(1,0)} M$ vectors. We can define these two forms $i \partial \bar{\partial} \rho_1$ and $i \partial \bar{\partial} \rho_2$ even at a CR singular point $p \in M$. 
These forms are the complex Hessian matrices of the defining equations. For our quadric $M$ they are the real and imaginary parts of the $(n+1) \times (n+1)$ complex matrix

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

(24)

where the variables are ordered as $(z_1, \ldots, z_n, w)$.

For $M$ to be Levi-flat, the quadratic form defined by $\tilde{A}$ has to be zero when restricted to the $n-1$ dimensional space spanned by $T_p^{(1,0)}M$ for every $p \in M_{CR}$. In other words for every $p \in M_{CR}$

$$v^* \tilde{A} v = 0, \quad \text{for all } v \in T_p^{(1,0)}M.$$

(25)

The space $T_p^{(1,0)}M$ is of dimension $n-1$, and furthermore, the vector $\frac{\partial}{\partial w}$ is not in $T_p^{(1,0)}M$. Therefore, $z^* A z = 0$ for $z \in \mathbb{C}^n$ in a subspace of dimension $n-1$.

Before we proceed let us note the following general fact about CR singular Levi-flat submanifolds.

**Lemma 3.1.** Suppose that $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is a Levi-flat connected real-analytic real codimension 2 submanifold, CR singular at the origin. Then there exists a germ of a complex analytic variety of complex dimension $n-1$ through the origin, contained in $M$.

**Proof.** Through each point of $M_{CR}$ there exists a germ of a complex variety of complex dimension $n-1$ contained in $M$. The set of CR points is dense in $M$. Take a sequence $p_k$ of CR points converging to the origin and take complex varieties of dimension $n-1$, $W_k \subset M$ with $p_k \in W_k$. A theorem of Fornæss (see Theorem 6.23 in [25] for a proof using the methods of Diederich and Fornæss [11]) implies that there exists a variety through $W \subset M$ with $0 \in W$ and of complex dimension at least $n-1$. □

Let us first concentrate on $n = 2$. When $n = 2$, $T^{(1,0)}M$ is one dimensional at CR points. Write

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

(26)

Note that $B$ is symmetric. A short computation shows that the vector field can be written as

$$\alpha \frac{\partial}{\partial w} + \beta_1 \frac{\partial}{\partial z_1} + \beta_2 \frac{\partial}{\partial z_2} = \alpha \frac{\partial}{\partial w} + \beta \cdot \frac{\partial}{\partial z},$$

(27)

where

$$\beta_1 = a_{21} \bar{z}_1 + a_{22} \bar{z}_2 + 2b_{12} z_1 + 2b_{22} z_2,$$

$$\beta_2 = -a_{11} \bar{z}_1 - a_{12} \bar{z}_2 - 2b_{11} z_1 - 2b_{12} z_2,$$

$$\alpha = a_{11} \bar{z}_1 \beta_1 + a_{21} \bar{z}_2 \beta_1 + a_{12} \bar{z}_1 \beta_2 + a_{22} \bar{z}_2 \beta_2.$$

(28)

Note that since the CR singular set is defined by $\beta_1 = \beta_2 = 0$, then $M_{CR}$ is dense in $M$. Thus we need to check that

$$[\beta^* \  \bar{\alpha}] \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \beta^* A \beta$$

is identically zero for $M$ to be Levi-flat.
If $A$ is the zero matrix, then $M$ is automatically Levi-flat. We diagonalize $B$ via $T$ into a diagonal matrix with ones and zeros on the diagonal. We obtain (recall $n = 2$) the submanifolds:

$$
w = z_1^2, \quad \text{or} \quad w = z_1^2 + z_2^2. \quad (30)$$

The first submanifold is of the form $M \times \mathbb{C}$ where $M \subset \mathbb{C}^2$ is a Bishop surface.

Let us from now on suppose that $A \neq 0$.

As $M$ is Levi-flat, then through each CR point $p = (z_p, w_p) \in M_{CR}$ we have a complex submanifold of dimension 1 in $M$. It is well-known that this submanifold is contained in the Segre variety (see also §4)

$$w = A(z, \bar{z}_p) + B(\bar{z}_p, z_p), \quad \bar{w}_p = \overline{A(z_p, z)} + \overline{B(z, z)}. \quad (31)$$

By Lemma 3.1, we obtain a complex variety $V \subset M$ of dimension one through the origin. Suppose without loss of generality that $V$ is irreducible. $V$ has to be contained in the Segre variety at the origin, in particular $w = 0$ on $V$. Therefore, to simplify notation, let us consider $V$ to be subvariety of $\{w = 0\}$. Denote by $\overline{V}$ the complex conjugate of $V$. Then as $V$ is irreducible, then $V \times \overline{V}$ is also irreducible (the smooth part of $V$ is connected and so the smooth part of $V \times \overline{V}$ is connected, see [30]). Hence, by complexifying, we have $A(z, \xi) + B(\xi, \bar{\xi}) = 0$ for all $z \in V$ and $\xi \in V$.

If $B \neq 0$, then setting $z = 0$, we have $B(\bar{\xi}, \xi) = 0$ on $V$. As $B$ is homogeneous and $V$ is irreducible, $V$ is a one dimensional complex line. If $B = 0$, then $A(z, \xi) = 0$ for $z, \xi \in V$ as mentioned above. We consider two cases. Suppose first that every $\sum_{j=1}^2 a_{ij} \bar{\xi}_j$ is identically zero for all $\xi \in V$ and $i = 1$ and $i = 2$. Then $V$ is contained in some complex line $\sum_{j=1}^2 a_{ij} \bar{\xi}_j = 0$. Suppose now that $A(z, \bar{\zeta}_s)$ is not identically zero for some $\zeta_s \in V$. Then $V$ is contained in the complex line $A(z, \bar{\zeta}_s) = 0$. This shows that $V$ is a complex line.

Thus as $A(z, \bar{z}) + B(z, \bar{z})$ is zero on a one dimensional linear subspace, we make this subspace $\{z_1 = 0\}$ and so each monomial in $A(z, \bar{z}) + B(z, \bar{z})$ is divisible by either $z_1$ or $\bar{z}_1$. Therefore, $A$ and $B$ are matrices of the form

$$\begin{bmatrix}
* & * \\
* & 0
\end{bmatrix}, \quad (32)$$

that is $a_{22} = 0$ and $b_{22} = 0$.

To normalize the pair $(A, B)$, we apply arbitrary invertible transformations $(T, \lambda) \in GL_n(\mathbb{C}) \times \mathbb{C}^*$ as

$$(A, B) \mapsto (\lambda T^{*}AT, \lambda T^{*}B\overline{T}). \quad (33)$$

Recall that we are assuming that $A \neq 0$. If $a_{21} = 0$ or $a_{12} = 0$, then $A$ is rank one and via a transformation $T$ of the form

$$z_1' = z_1, \quad z_2' = z_2 + cz_1 \quad \text{or} \quad z_2' = z_2, \quad z_1' = z_2 + cz_1 \quad (34)$$

and rescaling by nonzero $\lambda$, the matrix $A$ can be put in the form

$$\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}, \quad \text{or} \quad \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}. \quad (35)$$

The transformation $T$ and $\lambda$ must also be applied to $B$ and this could possibly make $b_{22} \neq 0$. However, we will show that we actually have $b_{22} = 0$. Thus $B = 0$ on $z_1 = 0$ still holds true.
Let us first focus on

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (36)$$

We apply the $T^{(1,0)}$ vector field we computed above. Only $a_{11}$ is nonzero in $A$. Therefore $\beta^* A \beta$, which must be identically zero, is

$$0 = \beta^* A \beta = \bar{\beta}_1 \beta_1 = (2b_{12} \bar{z}_1 + 2b_{22} \bar{z}_2)(2b_{12} \bar{z}_1 + 2b_{22} \bar{z}_2)$$

$$= 4(|b_{12}|^2 \bar{z}_1 \bar{z}_1 + |b_{22}|^2 \bar{z}_2 \bar{z}_2 + b_{12} \bar{b}_{22} \bar{z}_1 \bar{z}_2 + b_{12} b_{22} \bar{z}_1 \bar{z}_2). \quad (37)$$

This polynomial must be identically zero and hence all coefficients must be identically zero. So $b_{12} = 0$ and $b_{22} = 0$. In other words, only $b_{11}$ in $B$ can be nonzero, in which case we make it nonnegative via a diagonal $T$ to obtain the quadric

$$w = |z_1|^2 + \gamma \bar{z}_1^2, \quad \gamma \geq 0.$$

Next let us focus on

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad (39)$$

As above, we compute $\beta^* A \beta$:

$$0 = \beta^* A \beta = \bar{\beta}_1 \beta_2 = (2b_{12} \bar{z}_1 + 2b_{22} \bar{z}_2)(-\bar{z}_2 - 2b_{11} \bar{z}_1 - 2b_{12} \bar{z}_2)$$

$$= -2b_{12} \bar{z}_1 \bar{z}_2 - 2b_{22} \bar{b}_{11} \bar{z}_1 \bar{z}_2 - 4b_{11} b_{12} \bar{z}_1 \bar{z}_1 - 4b_{12} \bar{b}_{12} \bar{z}_1 \bar{z}_2 - 2b_{22} \bar{z}_2^2 - 4b_{22} \bar{b}_{12} \bar{z}_2 \bar{z}_2. \quad (40)$$

Again, as this polynomial must be identically zero, all coefficients must be zero. Hence $b_{12} = 0$ and $b_{22} = 0$. Again only $b_{11}$ is left possibly nonzero.

Suppose that $b_{11} \neq 0$. Then let $s$ be such that $b_{11} s^2 = 1$, and let $\bar{t} = \frac{1}{s}$. The matrix $T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is such that $T^* A T = A$ and $T^* B T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. If $b_{11} = 0$, we have $B = 0$. Therefore we have obtained two distinct possibilities for $B$, and thus the two submanifolds

$$w = \bar{z}_1 \bar{z}_2, \quad \text{or} \quad w = \bar{z}_1 \bar{z}_2 + \bar{z}_1^2. \quad (41)$$

We emphasize that after $A$ is normalized by a transformation of the form $(34)$, only one coordinate change is needed to normalize $b_{11}$ and this coordinate change preserves $A$. Both are required in a reduction proof for higher dimensions.

We have handled the rank one case. Next we focus on the rank two case, that is $a_{21} \neq 0$ and $a_{12} \neq 0$ (recall $a_{22} = 0$). We normalize (rescale) $A$ to have $a_{12} = 1$ and take

$$A = \begin{bmatrix} a_{11} & 1 \\ a_{21} & 0 \end{bmatrix}. \quad (42)$$

Again, let us compute $\beta^* A \beta$. In the computation for the rank 2 case, recall that we have not done any normalization other than rescaling, so we can safely still assume that $b_{22} = 0$.

$$0 = \beta^* A \beta = a_{11} \bar{\beta}_1 \beta_1 + \bar{\beta}_1 \beta_2 + a_{21} \bar{\beta}_1 \bar{\beta}_2$$

$$= a_{11}(\bar{a}_{21} \bar{z}_1 + 2b_{12} \bar{z}_1)(\bar{a}_{21} \bar{z}_1 + 2b_{12} \bar{z}_1) + (\bar{a}_{21} \bar{z}_1 + 2b_{12} \bar{z}_1)(-a_{11} \bar{z}_1 - \bar{z}_2 - 2b_{11} \bar{z}_1 - 2b_{12} \bar{z}_2)$$

$$+ a_{21}(\bar{a}_{11} \bar{z}_1 - \bar{z}_2 - 2b_{11} \bar{z}_1 - 2b_{12} \bar{z}_2)(\bar{a}_{21} \bar{z}_1 + 2b_{12} \bar{z}_1)$$

$$= (-4|b_{12}|^2 - |a_{21}|^2) \bar{z}_1 \bar{z}_2 + (\text{other terms}). \quad (43)$$

All coefficients must be zero. So $a_{21} = 0$, and $A$ would not be rank 2.
Let us now focus on \( n > 2 \). First let us suppose that \( A = 0 \). Then as before \( M \) is automatically \( n \)-Levi-flat and by diagonalizing \( B \) we obtain the \( n \) distinct submanifolds:

\[
  w = z_1^2, \\
  w = z_1^2 + \bar{z}_2^2, \\
  \vdots \\
  w = z_1^2 + \bar{z}_2^2 + \cdots + \bar{z}_n^2.
\]

(44)

Thus suppose from now on that \( A \neq 0 \). As before we have an irreducible \( n-1 \) dimensional variety \( V \subset M \) through the origin, such that \( w = 0 \) and \( A(z, \bar{z}) + B(\bar{z}, \bar{z}) = 0 \) on \( V \).

We wish to show that \( A(z, \bar{z}) + B(\bar{z}, \bar{z}) = 0 \) on an \( n-1 \) dimensional linear subspace. For any \( \xi \in V \) we obtain \( A(z, \bar{\xi}) + B(\bar{\xi}, \bar{\xi}) = 0 \) for all \( z \in V \). If \( V \) is contained in the kernel of the matrix \( A^* \), then we have that \( V \) is a linear subspace of dimension \( n-1 \). So suppose that \( \bar{\xi} \) is not in the kernel of the matrix \( A^* \). Then for a fixed \( \bar{\xi} \) we obtain a linear equation \( A(z, \bar{\xi}) + B(\bar{\xi}, \bar{\xi}) = 0 \) for \( z \in V \).

Therefore, as \( A(z, \bar{z}) + B(\bar{z}, \bar{z}) \) needs to be zero on an \( n-1 \) dimensional subspace we can just make this \( \{ z_1 = 0 \} \) and so each monomial is divisible by either \( z_1 \) or \( \bar{z}_1 \). Therefore, \( A \) and \( B \) is of the form

\[
\begin{bmatrix}
  * & * & \ldots & * \\
  * & 0 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  * & 0 & \ldots & 0
\end{bmatrix}
\]

(45)

that is, only first column and first row are nonzero. We normalize \( A \) via

\[
(A, B) \mapsto (\lambda T^* A T, \lambda \overline{T^*} B \overline{T}),
\]

(46)
as before. We use column operations on all but the first column to make all but the first two columns have nonzero elements. Similarly we can do row operations on all but the first two rows and to make all but first three rows nonzero. That is \( A \) has the form

\[
\begin{bmatrix}
  * & * & 0 & \ldots & 0 \\
  * & 0 & 0 & \ldots & 0 \\
  * & 0 & 0 & \ldots & 0 \\
  0 & 0 & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & 0
\end{bmatrix}
\]

(47)

By Lemma 2.2 setting \( z_3 = \cdots = z_n = 0 \) we obtain a Levi-flat submanifold where the matrix corresponding to \( A \) is the principal \( 2 \times 2 \) submatrix of \( A \). This submatrix cannot be of rank 2 and hence either \( a_{12} = 0 \) or \( a_{21} = 0 \). If \( a_{21} = 0 \) and \( a_{12} \neq 0 \), then setting \( z_2 = z_3 \), and \( z_4 = \cdots = z_n = 0 \) we again must have a rank one matrix and therefore \( a_{31} = 0 \).

Therefore, if \( a_{12} \neq 0 \) then all but \( a_{11} \) and \( a_{12} \) are zero. If \( a_{12} = 0 \), then via a further linear map not involving \( z_1 \) we can ensure that \( a_{31} = 0 \). In particular, \( A \) is of rank 1 and can only be nonzero in the principal \( 2 \times 2 \) submatrix. At this point \( B \) is still of the form (45).

Via a linear change of coordinates in the first two variables, the principal \( 2 \times 2 \) submatrix of \( A \) can be normalized into one of the 2 possible forms

\[
\begin{bmatrix}
  1 & 0 \\
  0 & 0
\end{bmatrix}, \quad \text{or} \quad
\begin{bmatrix}
  0 & 1 \\
  0 & 0
\end{bmatrix}.
\]

(48)
Recall that $A = 0$ was already handled.

Via the 2 dimensional computation we obtain that $b_{22} = b_{12} = b_{21} = 0$. We use a linear map in $z_1$ and $z_2$ to also normalize the principal $2 \times 2$ matrix of $B$, so that the submanifold restricted to $(z_1, z_2, w)$ is in one of the normal forms $B.\gamma$, C.0, or C.1.

Finally we need to show that all entries of $B$ other than $b_{11}$ are zero. As we have done a linear change of coordinates in $z_1$ and $z_2$, $B$ may not be in the form (45), but we know $b_{jk} = 0$ as long as $j > 2$ and $k > 2$.

Now fix $k = 3, \ldots, n$. Restrict to the submanifold given by $z_1 = \lambda z_2$ for $\lambda = 1$ or $\lambda = -1$, and $z_j = 0$ for all $j = 3, \ldots, n$ except for $j = k$. In the variables $(z_2, z_k, w)$, we obtain a Levi-flat submanifold where the matrix corresponding to $A$ is $[\lambda 0 0]$. The matrix corresponding to $B$ is

\[
\begin{bmatrix}
  b_{11} & b_{1k} + \lambda b_{2k} \\
  b_{1k} + \lambda b_{2k} & 0
\end{bmatrix}.
\]  

(49)

Via the 2 dimensional calculation we have $b_{1k} + \lambda b_{2k} = 0$. As this is true for $\lambda = 1$ and $\lambda = -1$, we get that $b_{1k} = b_{2k} = 0$.

We have proved the following classification result. It is not difficult to see that the submanifolds in the list are biholomorphically inequivalent by Proposition 2.1. The ranks of $A$ and $B$ are invariants. It is obvious that the $A$ matrix of $B.\gamma$ and C.x submanifolds are inequivalent. Therefore, it is only necessary to directly check that $B.\gamma$ are inequivalent for different $\gamma \geq 0$, which is easy.

**Lemma 3.2.** If $M$ defined in $(z, w) \in \mathbb{C}^n \times \mathbb{C}, n \geq 1$, by

\[
w = A(z, \bar{z}) + B(\bar{z}, \bar{z})
\]  

(50)

is Levi-flat, then $M$ is biholomorphic to one and exactly one of the following:

\[
\begin{align*}
(A.1) & \quad w = \bar{z}_1^2, \\
(A.2) & \quad w = \bar{z}_1^2 + \bar{z}_2^2, \\
& \quad \vdots \\
(A.n) & \quad w = \bar{z}_1^2 + \bar{z}_2^2 + \cdots + \bar{z}_n^2, \\
(B.\gamma) & \quad w = |z_1|^2 + \gamma \bar{z}_1^2, \quad \gamma \geq 0, \\
(C.0) & \quad w = \bar{z}_1 z_2, \\
(C.1) & \quad w = \bar{z}_1 z_2 + \bar{z}_1^2.
\end{align*}
\]  

(51)

The normalizing transformation used above is linear.

**Lemma 3.3.** If $M$ defined by

\[
w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + O(3)
\]  

(52)

is Levi-flat at all points where $M$ is CR, then the quadric

\[
w = A(z, \bar{z}) + B(\bar{z}, \bar{z})
\]  

(53)

is also Levi-flat.
Proof. Write $M$ as

$$w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + r(z, \bar{z}), \quad (54)$$

where $r$ is $O(3)$.

Let $A$ be the matrix giving the quadratic form $A(z, \bar{z})$ as before. The Levi-map is given by taking the $n \times n$ matrix

$$L = L(p) = A + \left[ \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} \right]_{j,k} \quad (55)$$

and applying it to vectors in $\pi(T^{(1,0)}M)$, where $\pi$ is the projection onto the $\{w = 0\}$ plane. That is we parametrize $M$ by the $\{w = 0\}$ plane, and work there as before.

Let

$$a_j = -\bar{A}_{z_j} - \bar{B}_{\bar{z}_j} - \bar{r}_{z_j},$$

$$b = \bar{A}_{z_1} + \bar{B}_{\bar{z}_1} + \bar{r}_{z_1},$$

$$c = a_j(A_{z_1} + B_{\bar{z}_1} + r_{z_1}) + b(A_{z_j} + B_{\bar{z}_j} + r_{z_j}). \quad (56)$$

Then for $j = 2, \ldots, n$, we write the $T^{(1,0)}$ vector fields as

$$X_j = a_j \frac{\partial}{\partial z_1} + b \frac{\partial}{\partial \bar{z}_j} + c \frac{\partial}{\partial w}. \quad (57)$$

Hence $a_j \frac{\partial}{\partial z_1} + b \frac{\partial}{\partial \bar{z}_j}$ are the vector fields in $\pi(T^{(1,0)}M)$.

Notice that $a_j$, $b$, and $c$ vanish at the origin, and furthermore that if we take the linear terms of $a_j$, $b$, and the quadratic terms in $c$, that is

$$\tilde{a}_j = -\bar{A}_{z_j} - \bar{B}_{\bar{z}_j},$$

$$\tilde{b} = \bar{A}_{z_1} + \bar{B}_{\bar{z}_1},$$

$$\tilde{c} = \tilde{a}_j(A_{z_1} + B_{\bar{z}_1}) + \tilde{b}(A_{z_j} + B_{\bar{z}_j}), \quad (58)$$

then away from the CR singular set of the quadric

$$\tilde{X}_j = \tilde{a}_j \frac{\partial}{\partial z_1} + \tilde{b} \frac{\partial}{\partial \bar{z}_j} + \tilde{c} \frac{\partial}{\partial w} \quad (59)$$

span the $T^{(1,0)}$ vector fields on the quadric $w = A(z, \bar{z}) + B(\bar{z}, \bar{z})$.

Since $M$ is Levi-flat, then we have that

$$\pi(X_j)^* L \pi(X_j) = 0. \quad (60)$$

The linear terms in $z$ and $\bar{z}$ in the expression $\pi(X_j)^* L \pi(X_j)$ are precisely

$$\pi(\tilde{X}_j)^* A \pi(\tilde{X}_j). \quad (61)$$

As this expression is identically zero, the quadric $w = A(z, \bar{z}) + B(\bar{z}, \bar{z})$ is Levi-flat. \qed

4. Quadratic Levi-flat submanifolds and their Segre varieties

A very useful invariant in CR geometry is the Segre variety. Suppose that a real-analytic variety $X \subset \mathbb{C}^N$ is defined by

$$\rho(z, \bar{z}) = 0, \quad (62)$$

where $\rho$ is a real-analytic real vector-valued with $p \in X$. Suppose that $\rho$ converges on some polydisc $\Delta$ centered at $p$. We complexify and treat $z$ and $\bar{z}$ as independent variables, and
the power series of $\rho$ at $(p, \bar{p})$ converges on $\Delta \times \Delta$. The Segre variety at $p$ is then defined as the variety

$$Q_p = \{ z \in \Delta : \rho(z, \bar{p}) = 0 \}. \quad (63)$$

Of course the variety depends on the defining equation itself and the polydisc $\Delta$. For $\rho$ it is useful to take the defining equation or equations that generate the ideal of the complexified $X$ in $\mathbb{C}^N \times \mathbb{C}^N$ at $p$. If $\rho$ is polynomial we take $\Delta = \mathbb{C}^N$.

It is well-known that any irreducible complex variety that lies in $X$ and goes through the point $p$ also lies in $Q_p$. In case of Levi-flat submanifolds we generally get equality as germs. For example, for the CR Levi-flat submanifold $M$ given by

$$\text{Im} z_1 = 0, \quad \text{Im} z_2 = 0, \quad (64)$$

the Segre variety $Q_0$ through the origin is precisely $\{ z_1 = z_2 = 0 \}$, which happens to be the unique complex variety in $M$ through the origin.

Let us take the Levi-flat quadric

$$w = A(z, \bar{z}) + B(\bar{z}, \bar{z}). \quad (65)$$

As we want to take the generating equations in the complexified space we also need the conjugate

$$\bar{w} = \bar{A}(\bar{z}, z) + \bar{B}(z, z). \quad (66)$$

The Segre variety is then given by

$$w = 0, \quad B(z, z) = 0. \quad (67)$$

Through any CR singular point of a real-analytic Levi-flat $M$ there is a complex variety of dimension $n - 1$ that is the limit of the leaves of the Levi-foliation of $M_{CR}$, via Lemma $3.1$. Let us take all possible such limits, and call their union $Q'_p$. Notice that there could be other complex varieties in $M$ through $p$ of dimension $n - 1$. Note that $Q'_p \subset Q_p$.

Let us write down and classify the Segre varieties for all the quadric Levi-flat submanifolds in $\mathbb{C}^{n+1}$:

<table>
<thead>
<tr>
<th>Type</th>
<th>Segre variety $Q_0$</th>
<th>$Q_0$ singular?</th>
<th>dim$_C Q_0$</th>
<th>$Q_0 \subset M$?</th>
<th>$Q'_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.1</td>
<td>$w = 0, z_1^2 = 0$</td>
<td>no</td>
<td>$n - 1$</td>
<td>yes</td>
<td>$Q_0$</td>
</tr>
<tr>
<td>A.$k$</td>
<td>$w = 0, z_1^2 + \cdots + z_k^2 = 0$</td>
<td>yes</td>
<td>$n - 1$</td>
<td>yes</td>
<td>$Q_0$</td>
</tr>
<tr>
<td>B.0</td>
<td>$w = 0$</td>
<td>no</td>
<td>$n$</td>
<td>no</td>
<td>$w = 0, z_1 = 0$</td>
</tr>
<tr>
<td>B.$\gamma$, $\gamma &gt; 0$</td>
<td>$w = 0, z_1^2 = 0$</td>
<td>no</td>
<td>$n - 1$</td>
<td>yes</td>
<td>$Q_0$</td>
</tr>
<tr>
<td>C.0</td>
<td>$w = 0$</td>
<td>no</td>
<td>$n$</td>
<td>no</td>
<td>$w = 0, z_1 = 0$</td>
</tr>
<tr>
<td>C.$1$</td>
<td>$w = 0, z_1^2 = 0$</td>
<td>no</td>
<td>$n - 1$</td>
<td>yes</td>
<td>$Q'_0$</td>
</tr>
</tbody>
</table>

The submanifold C.0 also contains the complex variety $\{ w = 0, z_2 = 0 \}$, but this variety is transversal to the leaves of the foliation, and so cannot be in $Q'_0$.

Notice that in the cases A.$k$ for all $k$, B.$\gamma$ for $\gamma > 0$, and C.$1$, the variety $Q_0$ actually gives the complex variety $Q'_0$ contained in $M$ through the origin. In these cases, the variety is nonsingular only in the set theoretic sense. Scheme-theoretically the variety is always at least a double line or double hyperplane in general.
5. The CR singularity of Levi-flats quadrics

Let us study the set of CR singularities for Levi-flat quadrics. The following proposition is well-known.

**Proposition 5.1.** Let $M \subset \mathbb{C}^{n+1}$ be given by

$$w = \rho(z, \bar{z})$$

(68)

where $\rho$ is $O(2)$, and $M$ is not a complex submanifold. Then the set $S$ of CR singularities of $M$ is given by

$$S = \{(z, w) : \bar{\partial}\rho = 0, w = \rho(z, \bar{z})\}.$$ 

(69)

**Proof.** In codimension 2, a real submanifold is either CR singular, complex, or generic. A submanifold is generic if $\bar{\partial}$ of all the defining equations are pointwise linearly independent (see [2]). As $M$ is not complex, to find the set of CR singularities, we find the set of points where $M$ is not generic. We need both defining equations for $M$,

$$w = \rho(z, \bar{z}), \quad \text{and} \quad \bar{w} = \rho(z, \bar{z}).$$

(70)

As the second equation always produces a $d\bar{w}$ while the first does not, the only way that the two can be linearly dependent is for the $\bar{\partial}$ of the first equation to be zero. In other words $\bar{\partial}\rho = 0$. □

Let us compute and classify the CR singular sets for the CR singular Levi-flat quadrics.

<table>
<thead>
<tr>
<th>Type</th>
<th>CR singularity $S$</th>
<th>dim$\mathbb{R}$ S</th>
<th>CR structure of $S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A,$k$</td>
<td>$z_1 = 0, \ldots, z_k = 0, w = 0$</td>
<td>$2n - 2k$</td>
<td>complex</td>
</tr>
<tr>
<td>B.0</td>
<td>$z_1 = 0, w = 0$</td>
<td>$2n - 2$</td>
<td>complex</td>
</tr>
<tr>
<td>B,$\frac{1}{2}$</td>
<td>$z_1 + \bar{z}_1 = 0, w = 0$</td>
<td>$2n - 1$</td>
<td>Levi-flat</td>
</tr>
<tr>
<td>B,$\gamma &gt; 0, \gamma \neq \frac{1}{2}$</td>
<td>$z_1 = 0, w = 0$</td>
<td>$2n - 2$</td>
<td>complex</td>
</tr>
<tr>
<td>C.0</td>
<td>$z_2 = 0, w = 0$</td>
<td>$2n - 2$</td>
<td>complex</td>
</tr>
<tr>
<td>C.1</td>
<td>$z_2 + 2\bar{z}_1 = 0, w = \frac{-z_2^2}{4}$</td>
<td>$2n - 2$</td>
<td>Levi-flat</td>
</tr>
</tbody>
</table>

By Levi-flat we mean that $S$ is a Levi-flat CR submanifold in $\{w = 0\}$. There is a conjecture that a real subvariety that is Levi-flat at CR points has a stratification by Levi-flat CR submanifolds. This computation gives further evidence of this conjecture.


A CR Levi-flat submanifold $M \subset \mathbb{C}^n$ of codimension 2 has a certain canonical foliation defined on it with complex analytic leaves of real codimension 2 in $M$. The submanifold $M$ is locally equivalent to $\mathbb{R}^2 \times \mathbb{C}^{n-2}$, defined by

$$\text{Im } z_1 = 0, \quad \text{Im } z_2 = 0.$$ 

(71)

The leaves of the foliation are the submanifolds given by fixing $z_1$ and $z_2$ at a real constant. By foliation we always mean the standard nonsingular foliation as locally comes up in the implicit function theorem. This foliation on $M$ is called the Levi-foliation. It is obvious that the Levi-foliation on $M$ extends to a neighbourhood of $M$ as a nonsingular holomorphic foliation. The same is not true in general for CR singular submanifolds. We say that a smooth holomorphic foliation $\mathcal{L}$ defined in a neighborhood of $M$ is an extension of the Levi-foliation of $M_{CR}$, if $\mathcal{L}$ and the Levi-foliation have the same germs of leaves at each CR point.
of $M$. We also say that a smooth real-analytic foliation $\mathcal{L}$ on $M$ is an extension of the Levi-foliation on $M_{CR}$ if $\mathcal{L}$ and the Levi-foliation have the same germs of leaves at each CR point of $M$. In our situation (real-analytic), $M_{CR}$ is a dense and open subset of $M$. This implies that the leaves of $\mathcal{L}$ and $\mathcal{L}$ through a CR singular point are complex analytic submanifolds also contained in $M$. The latter could lead to an obvious obstruction to extension. First let us see what happens if the foliation of $M_{CR}$ is the restriction of a nonsingular holomorphic foliation of a whole neighbourhood of $M$.

The Bishop-like quadrics, that is $A.1$ and $B.\gamma$ in $\mathbb{C}^{n+1}$, have a Levi-foliation that extends as a holomorphic foliation to all of $\mathbb{C}^{n+1}$. That is because these submanifolds are of the form $N \times \mathbb{C}^{n-1}$. (72)

For submanifolds of the form (72) we can find normal forms using the well-developed theory of Bishop surfaces in $\mathbb{C}^2$.

**Proposition 6.1.** Suppose $M \subset \mathbb{C}^{n+1}$ is a real-analytic Levi-flat CR singular submanifold where the Levi-foliation on $M_{CR}$ extends near $p \in M$ to a nonsingular holomorphic foliation of a neighbourhood of $p$ in $\mathbb{C}^{n+1}$. Then at $p$, $M$ is locally biholomorphically equivalent to a submanifold of the form

$$N \times \mathbb{C}^{n-1}.$$  

(73)

where $N \subset \mathbb{C}^2$ is a CR singular submanifold of real dimension 2. Therefore if $M$ has a nondegenerate complex tangent, then it is Bishop-like, that is of type $A.1$ or $B.\gamma$.

Furthermore, two submanifolds of the form (73) are locally biholomorphically (resp. formally) equivalent if and only if the corresponding $N$s are locally biholomorphically (resp. formally) equivalent in $\mathbb{C}^2$.

**Proof.** We flatten the holomorphic foliation near $p$ so that in some polydisc $\Delta$, the leaves of the foliation are given by $\{q\} \times \mathbb{C}^{n-1} \cap \Delta$ for $q \in \mathbb{C}^2$. Let us suppose that $M$ is closed in $\Delta$. At any CR point of $M$, the leaf of the Levi-foliation agrees with the leaf of the holomorphic foliation and therefore the leaf that lies in $M$ agrees with a leaf of the form $\{q\} \times \mathbb{C}^{n-1} \cap \Delta$ as a germ and so $\{q\} \times \mathbb{C}^{n-1} \cap \Delta \subset M$. As $M_{CR}$ is dense in $M$, then $M$ is a union of sets of the form $\{q\} \times \mathbb{C}^{n-1} \cap \Delta$ and the first part follows.

It is classical that every Bishop surface (2 dimensional real submanifold of $\mathbb{C}^2$ with a nondegenerate complex tangent) is equivalent to a submanifold whose quadratic part is of the form $A.1$ or $B.\gamma$.

Finally, the proof that two submanifolds of the form (73) are equivalent if and only if the $N$s are equivalent is straightforward. \[\square\]

Not every Bishop-like submanifold is a cross product as above. In fact the Bishop invariant may well change from point to point. See §15. In such cases the foliation does not extend to a nonsingular holomorphic foliation of a neighbourhood.

Let us now focus on extending the Levi-foliation to $M$, and not to a neighbourhood of $M$. Let us prove a useful proposition about recognizing certain CR singular Levi-flats from the form of the defining equation. That is if the $r$ in the equation does not depend on $\bar{z}_2$ through $\bar{z}_n$.

**Proposition 6.2.** Suppose near the origin $M \subset \mathbb{C}^{n+1}$ is given by

$$w = r(z_1, \bar{z}_1, z_2, z_3, \ldots, z_n),$$

(74)
where $r$ is $O(2)$ and $\frac{\partial r}{\partial z_1} \neq 0$. Then $M$ is a CR singular Levi-flat submanifold and the Levi-foliation of $M_{CR}$ extends through the origin to a real-analytic foliation on $M$. Furthermore, there exists a real-analytic CR mapping $F: U \subset \mathbb{R}^2 \times \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n+1}$, $F(0) = 0$, which is a diffeomorphism onto its image $F(U) \subset M$.

Near 0, $M$ is the image of a CR mapping that is a diffeomorphism onto its image of the standard CR Levi-flat. The proposition also holds in two dimensions ($n = 1$), although in this case it is somewhat trivial.

Proof. As in [27], let us define the mapping $F$ by
\[
(x, y, \xi) \mapsto (x + iy, \xi, r(x + iy, x - iy, \xi)),
\]
where $\xi = (\xi_2, \ldots, \xi_n) \in \mathbb{C}^{n-1}$. Near points where $M$ is CR, this mapping is a CR diffeomorphism and hence $M$ must be Levi-flat. Furthermore, since $F$ is a diffeomorphism, it takes the Levi-foliation on $\mathbb{R}^2 \times \mathbb{C}^{n-1}$ to a foliation on $M$ near 0. $\square$

In fact, we make the following conclusion.

Lemma 6.3. Let $M \subset \mathbb{C}^{n+1}$ be a CR singular real-analytic Levi-flat submanifold of codimension 2 through the origin.

Then $M$ is a CR singular Levi-flat submanifold whose Levi-foliation of $M_{CR}$ extends through the origin to a nonsingular real-analytic foliation on $M$ if and only if there exists a real-analytic CR mapping $F: U \subset \mathbb{R}^2 \times \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n+1}$, $F(0) = 0$, which is a diffeomorphism onto its image $F(U) \subset M$.

Proof. One direction is easy and was used above. For the other direction, suppose that we have a foliation extending the Levi-foliation through the origin. Let us consider $M_{CR}$ an abstract CR manifold. That is a manifold $M_{CR}$ together with the bundle $T^{(0,1)}M_{CR} \subset \mathbb{C} \otimes TM_{CR}$. The extended foliation on $M$ gives a real-analytic subbundle $\mathcal{W} \subset TM$. Since we are extending the Levi-foliation, when $p \in M_{CR}$, then $\mathcal{W}_p = T_p^cM$, where $T^c_pM = J(T^c_pM)$ is the complex tangent space and $J$ is the complex structure on $\mathbb{C}^{n+1}$. Since $M_{CR}$ is dense in $M$, then $J\mathcal{W} = \mathcal{W}$ on $M$.

Define the real-analytic subbundle $\mathcal{V} \subset \mathbb{C} \otimes TM$ as
\[
\mathcal{V}_p = \{X + iJ(X) : X \in \mathcal{W}_p\}.
\]
At CR points $\mathcal{V}_p = T^{(0,1)}_pM$ (see for example [2] page 8). Then we can find vector fields $X^1, \ldots, X^{n-1}$ in $\mathcal{W}$ such that
\[
X^1, J(X^1), X^2, J(X^2), \ldots, X^{n-1}, J(X^{n-1})
\]
is a basis of $\mathcal{W}$ near the origin. Then the basis for $\mathcal{V}$ is given by
\[
X^1 + iJ(X^1), X^2 + iJ(X^2), \ldots, X^{n-1} + iJ(X^{n-1}).
\]
As the subbundle is integrable, we obtain that $(M, \mathcal{V})$ gives an abstract CR manifold, which at CR points agrees with $M_{CR}$. This manifold is Levi-flat as it is Levi-flat on a dense open set. As it is real-analytic it is embeddable and hence there exists a real-analytic CR diffeomorphism from a neighbourhood of $\mathbb{R}^2 \times \mathbb{C}^{n-1}$ to a neighbourhood of 0 in $M$ (as an abstract CR manifold). This is our mapping $F$. $\square$
The quadrics $A.k$, $k \geq 2$, defined by
\[ w = z_1^2 + \cdots + z_k^2, \tag{79} \]
contain the singular variety defined by $w = 0$, $z_1^2 + \cdots + z_k^2 = 0$, and hence the Levi-foliation cannot extend to a nonsingular foliation of the submanifold. The quadric $A.1$ does admit a holomorphic foliation, but other type $A.1$ submanifolds do not in general. For example, the submanifold
\[ w = z_1^2 + z_2^3 \tag{80} \]
is of type $A.1$ and the unique complex variety through the origin is $0 = z_1^2 + z_2^3$, which is singular. Therefore the foliation cannot extend to $M$.

7. Extending the Levi-foliation of C.x type submanifolds

Let us prove Theorem 1.2, that is, let us start with a type C.0 or C.1 submanifold and show that the Levi-foliation must extend real-analytically to all of $M$. Equivalently, let us show that the real analytic bundle $T^{(1,0)}M_{CR}$ extends to a real analytic subbundle of $\mathbb{C} \otimes TM$, taking real parts we obtain an involutive subbundle of $TM$ extending $T^cM_{CR} = \text{Re}(T^{(1,0)}M_{CR})$.

Proof of Theorem 1.2. Let $M$ be the submanifold given by
\[ w = \bar{z}_1 z_2 + \epsilon \bar{z}_2^2 + r(z, \bar{z}) \tag{81} \]
where $\epsilon = 0, 1$. Let us treat the $z$ variables as the parameters on $M$. Let $\pi$ be the projection onto the $\{w = 0\}$ plane, which is tangent to $M$ at 0 as a real $2n$-dimensional hyperplane. We will look at all the vectorfields on this plane $\{w = 0\}$. All vectors in $\pi(T^{(1,0)}M)$ can be written in terms of $\frac{\partial}{\partial z_j}$ for $j = 1, \ldots, n$.

The Levi-map is given by taking the $n \times n$ matrix
\[ L = L(p) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + \left[ \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} \right]_{j,k} (p) \tag{82} \]
to vectors $v \in \pi(T^{(1,0)}M)$ ($\pi$ is the projection) as $v^*Lv$. The excess term in $L$ vanishes at 0.

Notice that for $p \in M_{CR}$, $\pi(T^{(1,0)}_p M)$ is $n - 1$ dimensional. As $M$ is Levi-flat, then $v^*Lv$ vanishes for $v \in \pi(T^{(1,0)}_p M)$. Write the vector $v = (v_1, \ldots, v_n)^t$. The zero set of the function
\[ (z, v) \in \mathbb{C}^n \times \mathbb{C}^n \mapsto v^*L(z, \bar{z})v \tag{83} \]
contains a variety $V$ of real codimension 2 because of the form of $L$. That is, at $z = 0$, the only vectors $v$ such that $v^*Lv = 0$ are those where $v_1 = 0$ or $v_2 = 0$. So the codimension is at least 2. And we know that $v^*Lv$ vanishes for vectors in $\pi(T^{(1,0)}_p M)$ for $p \in M$ near 0, which is real codimension 2 at each $z$ corresponding to a CR point. Therefore, as a variety the restriction of $V$ to $M_{CR}$, that is, $V \cap (\pi(M_{CR}) \times \mathbb{C}^n)$ has a component that is equal to the real-analytic subbundle $\pi(T^{(1,0)}M_{CR})$. 


We show below that this subbundle extends past the CR singularity. The key point is to show that the restriction of $\pi(T^{(1,0)}(M_{CR}))$ extends to a smooth real-analytic submanifold of $T^{(1,0)}\mathbb{C}^n$. Write
\[
\varphi(z, v) = v_1 \bar{v}_2 + \sum a_{jk}(z)v_j \bar{v}_k
\] (84)
where $a_{jk}(0) = 0$. When $z = 0$
\[
d\varphi = \bar{v}_2 dv_1 + v_1 d\bar{v}_2 \quad \text{and} \quad d\bar{\varphi} = v_2 d\bar{v}_1 + \bar{v}_1 dv_2.
\] (85)
Therefore when $z = 0$ the only point where $d\varphi$ and $d\bar{\varphi}$ are linearly dependent is when $v_1 = v_2 = 0$. We have to take both $\varphi$ and $\bar{\varphi}$ as we have a complex equation and hence two real equations.

Note that $M_{CR}$ is dense in $M$. For each $z \in \pi(M_{CR})$, there is an $n - 1$ dimensional complex linear subspace of vectors on which $\varphi$ vanishes. Therefore, the limit set of these subspaces must be in the zero set of $\varphi$ at $z = 0$. The dimension of the limit set must be at least $n - 1$, and hence at least one entire subspace $\{z = 0, v_1 = 0\}$ or $\{z = 0, v_2 = 0\}$ must be in the limit set.

The branch of $V \cap (\mathbb{C}^n \times (\mathbb{C}^n \setminus \{v_1 = v_2 = 0\}))$ corresponding to the limit set above must be a complex submanifold near points where $z = 0$, and as it is the limit set then for $z \in \pi(M_{CR})$ near 0, it agrees with the set
\[
\{(z, v) : \pi(p) = z \in \pi(M_{CR}), v \in \pi(T^{(1,0)}p)\}.
\] (86)
Therefore this set extends as a smooth real-analytic submanifold near $z = 0$ away from the set $v_1 = v_2 = 0$.

Let us write down a basis for the plane distribution. Write $W_z = \{v : (z, v) \in W\}$. Above $z = 0$ the set $W_0$ is an $n - 1$ complex linear space. Find a basis $\omega_1^0, \omega_2^0, \ldots, \omega_{n-1}^0$ for this space such that for no $\omega_j^0$ are the first two components both 0.

As at each $(0, \omega_j^0)$, $W$ is a submanifold transversal to the set $\{0\} \times \mathbb{C}^n$, we find real-analytic functions $\omega_j(z)$ for $z$ near 0, such that $\omega_j(0) = \omega_j^0$ and $\omega_j(z) \in W_z$. We have a subbundle of $T^{(1,0)}\mathbb{C}^n$ with fibres $W_z$.

We lift the functions $\omega_j$ via $\pi$ to a subbundle of $\mathbb{C}\otimes TM$, let us call these $\tilde{\omega}_j$. Then consider the vector fields $w_j = 2 \text{Re} \tilde{\omega}_j = \bar{\omega}_j + \omega_j$. Above CR points $w_j$ is in $T^cM_{CR} \subset TM_{CR}$ and so tangent to $M$. We thus obtain an $n - 1$ dimensional subbundle of $TM$ that agrees with $T^cM_{CR}$ above CR points.

This plane distribution is involutive at CR points which are dense and hence everywhere. □

8. CR singular set of type C.x submanifolds

Let $M \subset \mathbb{C}^{n+1}$ be a codimension two Levi-flat CR singular submanifold that is an image of $\mathbb{R}^2 \times \mathbb{C}^{n-1}$ via a real-analytic CR map, and let $S \subset M$ be the CR singular set of $M$. In [27] it was proved that near a generic point of $S$ exactly one of the following is true:

(i) $S$ is Levi-flat submanifold of dimension $2n - 2$ and CR dimension $n - 2$.
(ii) $S$ is a complex submanifold of complex dimension $n - 1$ (real dimension $2n - 2$).
(iii) $S$ is Levi-flat submanifold of dimension $2n - 1$ and CR dimension $n - 1$.

We only have the above classification for a generic point of $S$, and $S$ need not be a CR submanifold everywhere. See [27] for examples.
If \( M \) is a Levi-flat CR singular submanifold and the Levi-foliation of \( M_{CR} \) extends to \( M \), then by Lemma 6.3 then at a generic point \( S \) has to be of one of the above types. A corollary of Theorem 1.2 is the following result.

**Corollary 8.1.** Suppose that \( M \subset \mathbb{C}^{n+1}, n \geq 2, \) is a real-analytic Levi-flat CR singular type C.1 or type C.0 submanifold. Let \( S \subset M \) denote the CR singular set. Then near the origin \( S \) is a submanifold of dimension \( 2n - 2 \), and at a generic point, \( S \) is either CR Levi-flat of dimension \( 2n - 2 \) (CR dimension \( n - 2 \)) or a complex submanifold of complex dimension \( n - 1 \).

Furthermore, if \( M \) is of type C.1, then at the origin \( S \) is a CR Levi-flat submanifold of dimension \( 2n - 2 \) (CR dimension \( n - 2 \)).

**Proof.** Let us take \( M \) to be given by

\[
w = \bar{z}_1z_2 + \epsilon \bar{z}_1^2 + r(z, \bar{z})
\]

where \( r \) is \( O(3) \) and \( \epsilon = 0 \) or \( \epsilon = 1 \).

By Proposition 5.1 the CR singular set is exactly where

\[
z_2 + \epsilon 2\bar{z}_1 + r_{z_1}(z, \bar{z}) = 0, \quad \text{and} \quad r_{\bar{z}_1}(z, \bar{z}) = 0 \quad \text{for all } j = 2, \ldots, n.
\]

By considering the real and imaginary parts of the first equation and applying the implicit function theorem the set \( \tilde{S} = \{ z : z_2 + \epsilon 2\bar{z}_1 + r_{z_1}(z, \bar{z}) = 0 \} \) is a real submanifold of real dimension \( 2n - 2 \) (real codimension 2 in \( M \)). Now \( S \subset \tilde{S} \), but as we saw above \( S \) is of dimension at least \( 2n - 2 \). Therefore \( S = \tilde{S} \) near the origin. The conclusion of the first part then follows from the classification above.

The stronger conclusion for C.1 submanifolds follows by noticing that when \( \epsilon = 1 \), the submanifold

\[
z_2 + 2\bar{z}_1 + r_{\bar{z}_1}(z, \bar{z}) = 0
\]

is CR and not complex at the origin. □

**9. Mixed-holomorphic submanifolds**

Let us study sets in \( \mathbb{C}^m \) defined by

\[
f(\bar{z}_1, z_2, \ldots, z_m) = 0,
\]

for a single holomorphic function \( f \) of \( m \) variables.

Such sets have much in common with complex varieties, since they are in fact complex varieties when \( \bar{z}_1 \) is treated as a complex variable. The distinction is that the automorphism group is different since we are interested in automorphisms that are holomorphic not mixed-holomorphic.

**Proposition 9.1.** If \( M \subset \mathbb{C}^m \) is a submanifold with a defining equation of the form \( f(\bar{z}_1, z_2, \ldots, z_m) = 0 \), where \( f \) is a holomorphic function that is not identically zero, then \( M \) is a real codimension 2 set and \( M \) is either a complex submanifold or a Levi-flat submanifold, possibly CR singular. Furthermore, if \( M \) is CR singular at \( p \in M \), and has a nondegenerate complex tangent at \( p \), then \( M \) has type A.\( \kappa \), C.0, or C.1 at \( p \).

**Proof.** Since the zero set of \( f \) is a complex variety in the \( (\bar{z}_1, z_2, \ldots, z_m) \) space, we get automatically that it is real codimension 2. We also have that as it is a submanifold, then it can be written as a graph of one variable over the rest.
Let \( m = n + 1 \) for convenience and suppose that \( M \subset \mathbb{C}^{n+1} \) is a submanifold through the origin. By factorization for a germ of holomorphic function and by the smoothness assumption on \( M \) we may assume that \( df(0) \neq 0 \). Call the variables \((z_1, \ldots, z_n, w)\) and write \( M \) as a graph. One possibility is that we write \( M \) as

\[
\bar{w} = \rho(z_1, \ldots, z_n),
\]

where \( \rho(0) = 0 \) and \( \rho \) has no linear terms. \( M \) is complex if \( \rho \equiv 0 \). Otherwise \( M \) is CR singular and we rewrite it as

\[
w = \bar{\rho}(\bar{z}_1, \ldots, \bar{z}_n).
\]

We notice that the matrix representing the Levi-map must be identically zero, so we must get Levi-flat. If there are any quadratic terms we obtain a type A.\( k \) submanifold.

Alternatively \( M \) can be written as

\[
w = \rho(\bar{z}_1, z_2, \ldots, z_n),
\]

with \( \rho(0) = 0 \). If \( \rho \) does not depend on \( \bar{z}_1 \) then \( M \) is complex. Assume that \( \rho \) depends on \( \bar{z}_1 \). If \( \rho \) has linear terms in \( \bar{z}_1 \), then \( M \) is CR. Otherwise it is a CR singular submanifold, and near non-CR singular points it is a generic codimension 2 submanifold. The CR singular set of \( M \) is defined by \( \frac{\partial \rho}{\partial \bar{z}_1} = 0 \).

Suppose that \( M \) is CR singular. That \( M \) is Levi-flat follows from Proposition 6.2. We can therefore normalize the quadratic term, after linear terms in \( z_2, \ldots, z_n \) are absorbed into \( w \).

If not all quadratic terms are zero, then we notice that we must have an A.\( k \), C.0, or C.1 type submanifold.

Let us now study normal forms for such sets in \( \mathbb{C}^2 \) and \( \mathbb{C}^m \), \( m \geq 3 \). First in two variables we can easily completely answer the question. This result is surely well-known and classical.

**Proposition 9.2.** If \( M \subset \mathbb{C}^2 \) is a submanifold with a defining equation of the form

\[
w = \bar{z}^d
\]

for \( d = 0, 1, 2, 3, \ldots \) where \( d \) is a local biholomorphic invariant of \( M \). If \( d = 0 \), \( M \) is complex, if \( d = 1 \) it is a CR totally-real submanifold, and if \( d \geq 2 \) then \( M \) is CR singular.

**Proof.** Write the submanifold as a graph of one variable over the other. Without loss of generality and after possibly taking a conjugate of the equation, we have

\[
w = f(\bar{z})
\]

for some holomorphic function \( f \). Assume \( f(0) = 0 \). If \( f \) is identically zero, then \( d = 0 \) and we are finished. If \( f \) is not identically zero, then it is locally biholomorphic to a positive power of the variable. We apply a holomorphic change of coordinates in \( z \), and the rest follows easily. \( \square \)

In three or more variables, if \( M \subset \mathbb{C}^{n+1}, n \geq 2 \), is a submanifold through the origin, then if the quadratic part is nonzero we have seen above that it can be a type A.\( k \), C.0, or C.1 submanifold. If the submanifold is the nondegenerate type C.1 submanifold, then we will show in the next section that \( M \) is biholomorphically equivalent to the quadric \( M_{C.1} \).

Before we move to C.1, let us quickly consider the mixed-holomorphic submanifolds of type A.\( n \). The submanifolds of type A.\( n \) in \( \mathbb{C}^{n+1} \) can in some sense be considered nondegenerate when talking about mixed-holomorphic submanifolds.
Proposition 9.3. If $M \subset \mathbb{C}^{n+1}$ is a submanifold of type $A.n$ at the origin of the form
\[ w = \bar{z}_1^2 + \cdots + \bar{z}_n^2 + r(\bar{z}) \] (96)
where $r \in O(3)$. Then $M$ is locally near the origin biholomorphically equivalent to the $A.n$ quadric
\[ w = \bar{z}_1^2 + \cdots + \bar{z}_n^2. \] (97)

Proof. The complex Morse lemma states that there is a local change of coordinates near the origin in just the $z$ variables such that
\[ \bar{z}_1^2 + \cdots + \bar{z}_n^2 + r_{\bar{z}}(z) \] (98)
is equivalent to $z_1^2 + \cdots + z_n^2$.
\[ \square \]

It is not difficult to see that the normal form for mixed-holomorphic submanifolds in $\mathbb{C}^{n+1}$ of type $A.k$, $k < n$, is equivalent to a local normal form for a holomorphic function in $n$ variables. Therefore for example the submanifold $w = \bar{z}_1^2 + \bar{z}_2^3$ is of type $A.1$ and is not equivalent to any quadric.

10. Formal normal form for certain $C.1$ type submanifolds I

In this section we prove the formal normal form in Theorem 1.3. That is, we prove that if $M \subset \mathbb{C}^{n+1}$ is defined by
\[ w = \bar{z}_1 z_2 + \bar{z}_2 z_1 + r(z_1, \bar{z}_1, z_2, z_3, \ldots, z_n), \] (99)
where $r$ is $O(3)$, then $M$ is Levi-flat and formally equivalent to
\[ w = \bar{z}_1 z_2 + \bar{z}_2^2. \] (100)

That $M$ is Levi-flat follows from Proposition 6.2.

Lemma 10.1. If $M \subset \mathbb{C}^{n+1}, n \geq 2$, is given by
\[ w = \bar{z}_1 z_2 + \bar{z}_2^2 + r(z_1, \bar{z}_1, z_2, z_3, \ldots, z_n) \] (101)
where $r$ is $O(3)$ formal power series then $M$ is formally equivalent to $M_{C.1}$ given by
\[ w = \bar{z}_1 z_2 + \bar{z}_2^2. \] (102)

In fact, the normalizing transformation can be of the form
\[ (z, w) = (z_1, \ldots, z_n, w) \mapsto (z_1, f(z, w), z_3, \ldots, z_n, g(z, w)), \] (103)
where $f$ and $g$ are formal power series.

Proof. Suppose that the normalization was done to degree $d - 1$, then suppose that
\[ w = \bar{z}_1 z_2 + \bar{z}_2^2 + r_1(z_1, \bar{z}_1, z_2, \ldots, z_n) + r_2(z_1, \bar{z}_1, z_2, \ldots, z_n), \] (104)
where $r_1$ is degree $d$ homogeneous and $r_2$ is $O(d+1)$. Write
\[ r_1(z_1, \bar{z}_1, z_2, \ldots, z_n) = \sum_{j=0}^{k} \sum_{|\alpha|+j=d} c_{j,\alpha} z_1^j \bar{z}_1^{\alpha}, \] (105)
where $k$ is the highest power of $\bar{z}_1$ in $r_1$, and $\alpha$ is a multiindex.

If $k$ is even, then use the transformation that replaces $w$ with
\[ w + \sum_{|\alpha|+k=d} c_{j,\alpha} w^{k/2} \bar{z}_1^{\alpha}. \] (106)
Let us look at the degree $d$ terms in
\[
(\bar{z}_1 z_2 + \bar{z}_1^2) + \sum_{|\alpha| + k = d} c_{j,\alpha}(\bar{z}_1 z_2 + \bar{z}_1^2)^{k/2} z^\alpha = \bar{z}_1 z_2 + \bar{z}_1^2 + r_1(z_1, \bar{z}_1, z_2, \ldots, z_n). \tag{107}
\]

We need not include $r_2$ as the terms are all degree $d + 1$ or more. After cancelling out the new terms on the left, we notice that the formal transformation removed all the terms in $r_1$ with a power $\bar{z}_1^k$ and replaced them with terms that have a smaller power of $\bar{z}_1$.

Next suppose that $k$ is odd. We use the transformation that replaces $z_2$ with
\[
z_2 - \sum_{|\alpha| + k = d} c_{j,\alpha} w^{(k-1)/2} z^\alpha. \tag{108}
\]

Let us look at the degree $d$ terms in
\[
\bar{z}_1 z_2 + \bar{z}_1^2 = \bar{z}_1 \left(z_2 - \sum_{|\alpha| + k = d} c_{j,\alpha} w^{(k-1)/2} z^\alpha\right) + \bar{z}_1^2 + r_1(z_1, \bar{z}_1, z_2, \ldots, z_n). \tag{109}
\]

For $\bar{z}_1$ even, we need not include $r_2$. Since we can assume that all terms in $r_1$ depend on $\bar{z}_1$, we are finished with degree $d$ terms after $k$ iterations of the above procedure.

\[\square\]

11. Convergence of normalization for certain C.1 type submanifolds

A key point in the computation below is the following natural involution for the quadric $M_{C.1}$. Notice that the map
\[
(z_1, z_2, \ldots, z_n, w) \mapsto (-\bar{z}_2 - z_1, z_2, \ldots, z_n, w) \tag{110}
\]
takes $M_{C.1}$ to itself. The involution simply replaces the $\bar{z}_1$ in the equation with $-z_2 - \bar{z}_1$. The way this involution is defined is by noticing that the equation $w = \bar{z}_1 z_2 + \bar{z}_1^2$ has generically two solutions for $\bar{z}_1$ keeping $z_2$ and $w$ fixed. In the same way we could define an involution on all type C.1 submanifolds of the form $w = \bar{z}_1 z_2 + \bar{z}_2 + r(\bar{z}_1, z_2, \ldots, z_n)$, although we will not require this construction.

We prove convergence via the following well-known lemma:

**Lemma 11.1.** Let $m_1, \ldots, m_N$ be positive integers. Suppose $T(z)$ is a formal power series in $z \in \mathbb{C}^N$. Suppose $T(t^{m_1} v_1, \ldots, t^{m_N} v_N)$ is a convergent power series in $t \in \mathbb{C}$ for all $v \in \mathbb{C}^N$. Then $T$ is convergent.

The proof is a standard application of the Baire category theorem and the Cauchy inequality. See [2] (Theorem 5.5.30, p. 153) where all $m_j$ are 1. For $m_j > 1$ we first change variables by setting $v_j = w_j^{m_j}$ and apply the lemma with $m_j = 1$.

The following lemma finishes the proof of Theorem 11.3. By absorbing any holomorphic terms into $w$, we assume that $r(z_1, 0, z_2, \ldots, z_n) \equiv 0$. In Lemma 10.1, we have also constructed a formal transformation that only changed the $z_2$ and $w$ coordinates, so it is enough to prove convergence in this case. Key points of this proof are that the right hand side of the defining equation for $M_{C.1}$ is homogeneous, and that we have a natural involution on $M_{C.1}$. 
Lemma 11.2. If $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is given by

$$w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(z_1, \bar{z}_1, z_2, z_3, \ldots, z_n)$$

(111)

where $r$ is $O(3)$ and convergent, and $r(z_1, 0, z_2, \ldots, z_n) \equiv 0$. Suppose that two formal power series $f(z, w)$ and $g(z, w)$ satisfy

$$g(z, \bar{z}_1 z_2 + \bar{z}_1^2) = z_1 f(z, \bar{z}_1 z_2 + \bar{z}_1^2) + \bar{z}_1^2 + r(z_1, \bar{z}_1, f(z, \bar{z}_1 z_2 + \bar{z}_1^2), z_3, \ldots, z_n).$$

(112)

Then $f$ and $g$ are convergent.

Proof. The equation (112) is true formally, treating $z_1$ and $\bar{z}_1$ as independent variables. Notice that (112) has one equation for 2 unknown functions.

We now use the involution on $M_{C,1}$ to create a system that we can solve uniquely. We replace $\bar{z}_1$ with $-z_2 - \bar{z}_1$. We leave $z_1$ untouched (treating as an independent variable). We obtain an identity in formal power series:

$$g(z, \bar{z}_1 z_2 + \bar{z}_1^2) = (-z_2 - \bar{z}_1)f(z, \bar{z}_1 z_2 + \bar{z}_1^2) + (-z_2 - \bar{z}_1)^2$$

$$+ r(z_1, (-z_2 - \bar{z}_1), f(z, \bar{z}_1 z_2 + \bar{z}_1^2), z_3, \ldots, z_n).$$

(113)

The formal series $\xi = f(z, \bar{z}_1 z_2 + \bar{z}_1^2)$ and $\omega = g(z, \bar{z}_1 z_2 + \bar{z}_1^2)$ are solutions of the system

$$\omega = \bar{z}_1 \xi + \bar{z}_1^2 + r(z_1, \bar{z}_1, \xi, z_3, \ldots, z_n),$$

(114)

$$\omega = (-z_2 - \bar{z}_1)\xi + (-z_2 - \bar{z}_1)^2 + r(z_1, (-z_2 - \bar{z}_1), \xi, z_3, \ldots, z_n).$$

(115)

We next replace $z_j$ with $t z_j$ and $\bar{z}_1$ with $t \bar{z}_1$ for $t \in \mathbb{C}$. Because $\bar{z}_1 z_2 + \bar{z}_1^2$ is homogeneous of degree 2, we obtain that for every $(z_1, \bar{z}_1, z_2, \ldots, z_n) \in \mathbb{C}^{n+1}$ the formal series in $t$ given by $\xi(t) = f(tz, t^2(\bar{z}_1 z_2 + \bar{z}_1^2)), \omega(t) = g(tz, t^2(\bar{z}_1 z_2 + \bar{z}_1^2))$ are solutions of the system

$$\omega = tz_1 \xi + t^2 \bar{z}_1^2 + r(tz_1, t \bar{z}_1, \xi, z_3, \ldots, z_n),$$

(116)

$$\omega = t(-z_2 - \bar{z}_1)\xi + t^2(-z_2 - \bar{z}_1)^2 + r(tz_1, t(-z_2 - \bar{z}_1), \xi, z_3, \ldots, z_n).$$

(117)

We eliminate $\omega$ to obtain an equation for $\xi$:

$$t(2\bar{z}_1 + z_2)(\xi - t z_2) = r(tz_1, t(-z_2 - \bar{z}_1), \xi, z_3, \ldots, z_n) - r(tz_1, t \bar{z}_1, \xi, z_3, \ldots, z_n).$$

(118)

We now treat $\xi$ as a variable and we have a holomorphic (convergent) equation. The right hand size must be divisible by $t(2\bar{z}_1 + z_2)$: It is divisible by $t$ since $r$ was divisible by $\bar{z}_1$. It is also divisible by $2\bar{z}_1 + z_2$ as setting $z_2 = -2\bar{z}_1$ makes the right hand side vanish. Therefore,

$$\xi - t z_2 = \frac{r(tz_1, t(-z_2 - \bar{z}_1), \xi, z_3, \ldots, z_n) - r(tz_1, t \bar{z}_1, \xi, z_3, \ldots, z_n)}{t(2\bar{z}_1 + z_2)},$$

(119)

where the right hand side is a holomorphic function (that is, a convergent power series) in $z_1, \bar{z}_1, z_2, \ldots, z_n, t$. For any fixed $z_1, \bar{z}_1, z_2, \ldots, z_n$, we solve for $\xi$ in terms of $t$ via the implicit function theorem, and we obtain that $\xi$ is a holomorphic function of $t$. The power series of $\xi$ is given by $\xi(t) = f(tz, t^2(\bar{z}_1 z_2 + \bar{z}_1^2))$.

Let $v \in \mathbb{C}^{n+1}$ be any nonzero vector. Via proper choice of $z_1, \bar{z}_1, z_2, \ldots, z_n$ (still treating $\bar{z}_1$ and $z_1$ as independent variables) we write $v = (z, \bar{z}_1 z_2 + \bar{z}_1^2)$. We apply the above argument to $\xi(t) = f(tv_1, \ldots, tv_n, t^2v_{n+1})$, and $\xi(t)$ converges as a series in $t$. As we get convergence for every $v \in \mathbb{C}^{n+1}$ we obtain that $f$ converges by Lemma 11.1. Once $f$ converges, then via (116) we obtain that $g(tv_1, \ldots, tv_n, t^2v_{n+1})$ converges as a series in $t$ for all $v$, and hence $g$ converges. \qed
12. Automorphism group of the C.1 quadric

With the normal form achieved in previous sections, let us study the automorphism group of the C.1 quadric in this section. We will again use the mixed-holomorphic involution that is obtained from the quadric.

We study the local automorphism group at the origin. That is the set of germs at the origin of biholomorphic transformations taking $M$ to $M$ and fixing the origin.

First we look at the linear parts of automorphisms. We already know that the linear term of the last component only depends on $w$. For $M_{C.1}$ we can say more about the first two components.

**Proposition 12.1.** Let $(F, G) = (F_1, \ldots, F_n, G)$ be a formal invertible or biholomorphic automorphism of $M_{C.1} \subset \mathbb{C}^{n+1}$, that is the submanifold of the form

$$w = \bar{z}_1 z_2 + \bar{z}_2^2.$$  \hfill (120)

Then $F_1(z, w) = az_1 + \alpha w + O(2)$, $F_2(z, w) = \bar{a}z_2 + \beta w + O(2)$, and $G(z, w) = \bar{a}^2 w + O(2)$, where $a \neq 0$.

**Proof.** Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be such that $F_1(z, w) = a \cdot z + \alpha w + O(2)$ and $F_2(z, w) = b \cdot z + \beta w + O(2)$. Then from Proposition 2.1 we have

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \lambda \begin{bmatrix} a^* \\ b^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \hfill (121)$$

Therefore $\lambda a_1 b_2 = 1$, and $a_j b_k = 0$ for all $(j, k) \neq (1, 2)$. Similarly

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \lambda \begin{bmatrix} \bar{a}^t \\ \bar{b}^t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix}. \hfill (122)$$

Therefore $\lambda \bar{a}_1^2 = 1$, and $\bar{a}_j \bar{a}_k = 0$ for all $(j, k) \neq (1, 1)$. Putting these two together we obtain that $a_j = 0$ for all $j \neq 1$, and as $a_1 \neq 0$ we get $b_j = 0$ for all $j \neq 2$. As $\lambda$ is the reciprocal of the coefficient of $w$ in $G$, we are finished. \qed

**Lemma 12.2.** Let $M_{C.1} \subset \mathbb{C}^3$ be given by

$$w = \bar{z}_1 z_2 + \bar{z}_2^2.$$  \hfill (123)

Suppose that a local biholomorphism (resp. formal automorphism) $(F_1, F_2, G)$ transforms $M_{C.1}$ into $M_{C.1}$. Then $F_1$ depends only on $z_1$, and $F_2$ and $G$ depend only on $z_2$ and $w$.

**Proof.** Let us define a $(1,0)$ tangent vector field on $M$ by

$$Z = \frac{\partial}{\partial z_2} + \bar{z}_1 \frac{\partial}{\partial w}. \hfill (124)$$
Write $F = (F_1, F_2, G)$. $F$ must take $Z$ into a multiple of itself when restricted to $M_{C,1}$. That is on $M_{C,1}$ we have

$$\frac{\partial F_1}{\partial z_2} + \bar{z}_1 \frac{\partial F_1}{\partial w} = 0,$$

$$(125)$$

$$\frac{\partial F_2}{\partial z_2} + \bar{z}_1 \frac{\partial F_2}{\partial w} = \lambda,$$

$$(126)$$

$$\frac{\partial G}{\partial z_2} + \bar{z}_1 \frac{\partial G}{\partial w} = \lambda F_1(z, \bar{w}),$$

$$(127)$$

for some function $\lambda$. Let us take the first equation and plug in the defining equation for $M_1$.\[
\frac{\partial F_1}{\partial z_2}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) + \bar{z}_1 \frac{\partial F_1}{\partial w}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = 0.\]

$$(128)$$

This equation is true for all $z \in \mathbb{C}^2$, and so we may treat $z_1$ and $\bar{z}_1$ as independent variables. We have an involution on $M_{C,1}$ that takes $\bar{z}_1$ to $-z_2 - \bar{z}_1$. Therefore we also have

$$\frac{\partial F_1}{\partial z_2}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) + (-z_2 - \bar{z}_1) \frac{\partial F_1}{\partial w}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = 0.\]

$$(129)$$

This means that $\frac{\partial F_1}{\partial w}$ and therefore $\frac{\partial F_1}{\partial z_2}$ must be identically zero. That is, $F_1$ only depends on $z_1$.

We have that the following must hold for all $z$:

$$G(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = \bar{F}_1(\bar{z}_1) F_2(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) + (\bar{F}_1(\bar{z}_1))^2.\]

$$(130)$$

Again we treat $z_1$ and $\bar{z}_1$ as independent variables. We differentiate with respect to $z_1$:

$$\frac{\partial G}{\partial z_1}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = \bar{F}_1(\bar{z}_1) \frac{\partial F_2}{\partial z_1}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2)\]

$$(131)$$

We plug in the involution again to obtain

$$\frac{\partial G}{\partial z_1}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = \bar{F}_1(-z_2 - \bar{z}_1) \frac{\partial F_2}{\partial z_1}(z_1, z_2, \bar{z}_1 z_2 + \bar{z}_1^2).\]

$$(132)$$

Therefore as $F_1$ is not identically zero, then as before both $\frac{\partial F_2}{\partial z_1}$ and $\frac{\partial G}{\partial z_1}$ must be identically zero. \hfill \Box

**Lemma 12.3.** Take $M_{C,1} \subset \mathbb{C}^3$ given by

$$w = \bar{z}_1 z_2 + \bar{z}_1^2,$$

$$(133)$$

and let $(F_1, F_2, G)$ be a local automorphism at the origin. Then $F_1$ uniquely determines $F_2$ and $G$. Furthermore, given any invertible function of one variable $F_1$ with $F_1(0) = 0$, there exist unique $F_2$ and $G$ that complete an automorphism and they are determined by

$$F_2(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = \bar{F}_1(\bar{z}_1) + \bar{F}_1(-\bar{z}_1 - z_2),\]

$$(134)$$

$$G(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = -\bar{F}_1(\bar{z}_1) \bar{F}_1(-\bar{z}_1 - z_2).\]

We should note that the lemma also works formally. Given any formal $F_1$, there exist unique formal $F_2$ and $G$ satisfying the above property.
Proof. By Lemma 12.2, $F_1$ depends only on $z_1$ and $F_2$ and $G$ depend only on $z_2$ and $w$. We write the automorphism as a composition of the two mappings $(F_1(z_1), z_2, w)$ and $(z_1, F_2(z_2, w), G(z_2, w))$.

We plug the transformation into the defining equation for $M_{C,1}$.

$$G(z_2, ar{z}_1 z_2 + z_2^2) = \bar{F}_1(\bar{z}_1) F_2(z_2, \bar{z}_1 z_2 + z_2^2) + (F_1(\bar{z}_1))^2.$$  \hspace{1cm} (135)

We use the involution $(z_1, z_2) \mapsto (-\bar{z}_1 - z_2, z_2)$ which preserves $M_{C,1}$ and obtain a second equation

$$G(z_2, \bar{z}_1 z_2 + z_2^2) = \bar{F}_1(-\bar{z}_1 - z_2) F_2(z_2, \bar{z}_1 z_2 + z_2^2) + (\bar{F}_1(-\bar{z}_1 - z_2))^2.$$ \hspace{1cm} (136)

We eliminate $G$ and solve for $F_2$:

$$F_2(z_2, \bar{z}_1 z_2 + z_2^2) = \frac{(\bar{F}_1(-\bar{z}_1 - z_2))^2 - (\bar{F}_1(\bar{z}_1))^2}{\bar{F}_1(\bar{z}_1) - \bar{F}_1(-\bar{z}_1 - z_2)} = \bar{F}_1(\bar{z}_1) + \bar{F}_1(-\bar{z}_1 - z_2).$$ \hspace{1cm} (137)

Next we note that trivially, $F_2$ is unique if it exists: its difference vanishes on $M_{C,1}$.

If we suppose that $F_1$ is convergent, then just as before, substituting $z_2$ with $t z_2$ and $\bar{z}_1$ with $t \bar{z}_1$ are restricting to curves $(t z_2, t^2 w)$ for all $(z_2, w)$. Therefore if $F_2$ exists and $F_1$ is convergent, then $F_2$ is convergent.

Now we need to show the existence of the formal solution $F_2$. Notice that the right-hand side of (137) invariant under the involution. It suffices to show that any power series in $z_1, z_2$ that is invariant under the involution is a formal power series in $z_2$ and $\bar{z}_1 z_2 + z_2^2$. Let us treat $\xi = \bar{z}_1$ as an independent variable. The original involution becomes a holomorphic involution in $\xi, z_2$:

$$\tau: \xi \rightarrow -\xi - z_2, \quad z_2 \rightarrow z_2.$$ \hspace{1cm} (138)

By a theorem of Noether we obtain a set of generators for the ring of invariants can be obtained by applying the averaging operation $R(f) = \frac{1}{2} (f + f \circ \tau)$ to all monomials in $\xi$ and $z_2$ of degree 2 or less. By direct calculation it is not difficult to see that $\xi, \xi z_2 + \xi^2$ generate the ring of invariants. Therefore any invariant power series in $z_2, \xi$ is a power series in $\xi, \xi z_2 + \xi^2$. This shows the existence of $F_2$. The existence of $G$ follows the same.

The equation for $G(z_2, \bar{z}_1 z_2 + z_2^2) = -\bar{F}_1(\bar{z}_1) F_1(-\bar{z}_1 - z_2)$ is obtained by plugging in the equation for $F_2$. Its existence, uniqueness, and convergence in case $F_1$ converges, follows exactly the same as for $F_2$.

\begin{Theorem} \textbf{12.4.} \textit{If $M \subset \mathbb{C}^{n+1}, n \geq 2$ is given by}

$$w = \bar{z}_1 z_2 + \bar{z}_1^2,$$ \hspace{1cm} (139)

and $(F_1, F_2, \ldots, F_n, G)$ is a local automorphism at the origin, then $F_1$ depends only on $z_1$, $F_2$ and $G$ depend only on $z_2$ and $w$, and $F_1$ completely determines $F_2$ and $G$ via (134). The mapping $(z_1, z_2, F_3, \ldots, F_n)$ has rank $n$ at the origin.

Furthermore, given any invertible function of one variable $F_1$ with $F_1(0) = 0$, and arbitrary holomorphic functions $F_3, \ldots, F_n$ with $F_j(0) = 0$, and such that $(z_1, z_2, F_3, \ldots, F_n)$ has rank $n$ at the origin, then there exist unique $F_2$ and $G$ that complete an automorphism.
\end{Theorem}

Proof. Let $(F_1, \ldots, F_n, G)$ be an automorphism. Then we have

$$G(z_1, \ldots, z_n, w) = \bar{F}_1(\bar{z}_1, \ldots, \bar{z}_n, \bar{w}) F_2(z_1, \ldots, z_n, w) + (\bar{F}_1(\bar{z}_1, \ldots, \bar{z}_n, \bar{w}))^2.$$ \hspace{1cm} (140)

Proposition 12.1 says that the linear terms in $G$ only depend on $w$, the linear terms of $F_1$ depend only on $z_1$ and $w$ and the linear terms of $F_2$ only depend on $z_2$ and $w$. 

Let us embed $M_{C,1} \subset \mathbb{C}^3$ into $M$ via setting $z_3 = \alpha_3 \bar{z}_2$, $\ldots$, $z_n = \alpha_n \bar{z}_2$, for arbitrary $\alpha_3, \ldots, \alpha_n$. Then we obtain

$$G(z_1, z_2, \alpha_3 \bar{z}_2, \ldots, \alpha_n \bar{z}_2, w) = \frac{F_1(\bar{z}_1, \bar{z}_2, \bar{\alpha}_3 \bar{z}_2, \ldots, \bar{\alpha}_n \bar{z}_2, \bar{w})}{F_2(z_2, \alpha_3 \bar{z}_2, \ldots, \alpha_n \bar{z}_2, \bar{w})} F_2(z_1, z_2, \alpha_3 \bar{z}_2, \ldots, \alpha_n \bar{z}_2, w) + \left(\frac{F_1(\bar{z}_1, \bar{z}_2, \bar{\alpha}_3 \bar{z}_2, \ldots, \bar{\alpha}_n \bar{z}_2, \bar{w})}{F_2(z_2, \alpha_3 \bar{z}_2, \ldots, \alpha_n \bar{z}_2, \bar{w})}\right)^2. \quad (141)$$

By noting what are the linear terms are, we notice that the above is the equation for an automorphism of $M_{C,1}$. Therefore by Lemma 12.2 we have

$$\frac{\partial F_1}{\partial w} = 0 \quad \text{and} \quad \frac{\partial F_2}{\partial z_1} = 0 \quad \text{and} \quad \frac{\partial G}{\partial z_1} = 0, \quad (142)$$

as that is true for all $\alpha_3, \ldots, \alpha_n$. Plugging in the defining equation for $M_{C,1}$ we obtain an equation that holds for all $z$ and we can treat $z$ and $\bar{z}$ independently. We plug in $z = 0$ to obtain

$$0 = \frac{F_1(\bar{z}_1, \bar{z}_2, \bar{\alpha}_3 \bar{z}_2, \ldots, \bar{\alpha}_n \bar{z}_2, 0)}{F_2(0, \ldots, 0, \bar{z}_1^2)} F_2(z_2, \alpha_3 \bar{z}_2, \ldots, \alpha_n \bar{z}_2, 0) + \left(\frac{F_1(\bar{z}_1, \bar{z}_2, \bar{\alpha}_3 \bar{z}_2, \ldots, \bar{\alpha}_n \bar{z}_2, 0)}{F_2(0, \ldots, 0, \bar{z}_1^2)}\right)^2. \quad (143)$$

Differentiating with respect to $\bar{\alpha}_j$ we obtain $\frac{\partial F_1}{\partial \bar{\alpha}_j} = 0$, for $j = 3, \ldots, n$. We set $\bar{\alpha}_j = 0$ in the equation, differentiate with respect to $\bar{z}_2$ and obtain that $\frac{\partial F_1}{\partial \bar{z}_2} = 0$. In other words $F_1$ is a function of $z_1$ only. We rewrite (141) by writing $F_1$ as a function of $z_1$ only and $F_2$ and $G$ as functions of $z_2, \ldots, z_n, w$, and we plug in $w = \bar{z}_1 z_2 + \bar{z}_1^2$.

$$G(z_2, \alpha_3 z_2, \ldots, \alpha_n z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = \frac{F_1(\bar{z}_1)}{F_2(z_2, \alpha_3 z_2, \ldots, \alpha_n z_2, \bar{z}_1 z_2 + \bar{z}_1^2)} F_2(z_2, \alpha_3 z_2, \ldots, \alpha_n z_2, \bar{z}_1 z_2 + \bar{z}_1^2) + \left(\frac{F_1(\bar{z}_1)}{F_2(z_2, \alpha_3 z_2, \ldots, \alpha_n z_2, \bar{z}_1 z_2 + \bar{z}_1^2)}\right)^2. \quad (144)$$

By Lemma 12.3, we know that $F_1$ now uniquely determines $F_2(z_2, \alpha_3 z_2, \ldots, \alpha_n z_2, w)$ and $G(z_2, \alpha_3 z_2, \ldots, \alpha_n z_2, w)$. These two functions therefore do not depend on $\alpha_3, \ldots, \alpha_n$, and in turn $F_2$ and $G$ do not depend on $z_3, \ldots, z_n$ as claimed. Furthermore $F_1$ does uniquely determine $F_2$ and $G$.

Finally since the mapping is a biholomorphism, and from what we know the linear parts of $F_1$, $F_2$, and $G$ are, it is clear that $(z_1, z_2, F_3, \ldots, F_n)$ is rank $n$.

The other direction follows by applying Lemma 12.3. We start with $F_1$, determine $F_2$ and $G$ as in 3 dimensions. Then adding $F_3, \ldots, F_n$ and the rank condition guarantees an automorphism.

13. Normal Form for Certain C.1 Type Submanifolds II

The goal of this section is to find the normal form for Levi-flat submanifolds $M \subset \mathbb{C}^{n+1}$ given by

$$w = \bar{z}_1 z_2 + \bar{z}_1^2 + \text{Re } f(z), \quad (145)$$

for a holomorphic $f(z)$ of order $O(3)$.

Since $f(z)$ can be absorbed into $w$ via a holomorphic transformation, the goal is really to prove the following theorem.

**Theorem 13.1.** Let $M \subset \mathbb{C}^{n+1}$ be a real-analytic Levi-flat given by

$$w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(z), \quad (146)$$

where
where \( r \) is \( O(3) \). Then \( M \) can be put into the \( M_{C,1} \) normal form
\[
w = \bar{z}_1 z_2 + \bar{z}^2_1,
\] (147)
by a convergent normalizing transformation.

Furthermore, if \( r \) is a polynomial and the coefficient of \( \bar{z}_1^3 \) in \( r \) is zero, then there exists an invertible polynomial mapping taking \( M_{C,1} \) to \( M \).

In Theorem 1.3 we have already shown that a submanifold of the form
\[
w = \bar{z}_1 z_2 + \bar{z}^2_1 + r(\bar{z}_1)
\] (148)
is necessarily Levi-flat and has the normal form \( M_{C,1} \). The first part of Theorem 13.1 will follow once we prove:

**Lemma 13.2.** If \( M \subset \mathbb{C}^{n+1} \) is given by
\[
w = \bar{z}_1 z_2 + \bar{z}^2_1 + r(\bar{z})
\] (149)
where \( r \) is \( O(3) \) and \( M \) is Levi-flat, then \( r \) depends only on \( \bar{z}_1 \).

**Proof.** First let us assume that \( n = 2 \). For \( p \in M_{CR} \), \( T_p^{(1,0)}M \) is one dimensional. The Levi-map is the matrix
\[
L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\] (150)
applied to the \( T^{(1,0)}M \) vectors. As \( M \) is Levi-flat, then the Levi-map has to vanish. The only vectors \( v \) for which \( v^* Lv = 0 \), are when either there is no \( \frac{\partial}{\partial z_1} \) component or no \( \frac{\partial}{\partial z_2} \) component. That is vectors of the form
\[
a \frac{\partial}{\partial z_1} + b \frac{\partial}{\partial w}, \quad \text{or} \quad a \frac{\partial}{\partial z_2} + b \frac{\partial}{\partial w}.
\] (151)
We apply these vectors to the defining equation and its conjugate and we obtain in the first case the equations
\[
b = 0, \quad 0 = a \left( \bar{z}_2 + 2z_1 + \frac{\partial r}{\partial z_1} \right) = 0.
\] (152)
This cannot be satisfied identically on \( M \) since this is supposed to be true for all \( z \), but \( a \) cannot be identically zero and the second factor in the second equation has only one nonholomorphic term, which is \( \bar{z}_2 \).

Let us try the second form and we obtain the equations
\[
b = a \bar{z}_1, \quad 0 = a \left( \frac{\partial r}{\partial z_2} \right) = 0.
\] (153)
Again \( a \) cannot be identically zero, and hence the second factor of the second equation \( \frac{\partial r}{\partial z_2} \) must be identically zero, which is possible only if \( r \) depends only on \( \bar{z}_1 \).

Finally, it is possible to pick \( b = \bar{z}_1 \) and \( a = 1 \), to obtain a \( T^{(1,0)} \) vector field
\[
\frac{\partial}{\partial z_2} + \bar{z}_1 \frac{\partial}{\partial w},
\] (154)
and therefore these submanifolds are necessarily Levi-flat.
Next suppose that \( n > 2 \). Notice that replacing \( z_k \) with \( \lambda_k \xi \) for \( k \geq 2 \) and then fixing \( \lambda_k \) for \( k \geq 2 \), we get
\[
w = \bar{z}_1 \lambda_2 \xi + \bar{z}_1^2 + r(\bar{z}_1, \bar{\lambda}_2 \xi, \ldots, \bar{\lambda}_n \xi).
\] (155)
By Lemma 2.2, we obtain a Levi-flat submanifold in \((z_1, \xi, w) \in \mathbb{C}^3\), and hence can apply the above reasoning to obtain that \( r(\bar{z}_1, \bar{\lambda}_2 \xi, \ldots, \bar{\lambda}_n \xi) \) does not depend on \( \bar{\xi} \). As this was true for any \( \lambda_k \)'s, we have that \( r \) can only depend on \( \bar{z}_1 \).

It is left to prove the claim about the polynomial normalizing transformation.

**Lemma 13.3.** Suppose \( M \subset \mathbb{C}^{n+1} \) is given by
\[
w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(\bar{z}_1)
\] (156)
where \( r \) is a polynomial that vanishes to fourth order. Then there exists an invertible polynomial mapping taking \( M_{C.1} \) to \( M \).

**Proof.** We will take a transformation of the form
\[
(z_1, z_2, w) \mapsto (z_1, z_2 + f(z_2, w), w + g(z_2, w)).
\] (157)
We are therefore trying to find polynomial \( f \) and \( g \) that satisfy
\[
\bar{z}_1 z_2 + \bar{z}_1^2 + g(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = \bar{z}_1 (\bar{z}_1 z_2 + \bar{z}_1^2 + f(z_2, \bar{z}_1 z_2 + \bar{z}_1^2)) + \bar{z}_1^2 + r(\bar{z}_1).
\] (158)
If we simplify we obtain
\[
g(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) - \bar{z}_1 f(z_2, \bar{z}_1 z_2 + \bar{z}_1^2) = r(\bar{z}_1).
\] (159)
Consider the involution \( S : (\bar{z}_1, z_2) \mapsto (-\bar{z}_1 - z_2, z_2) \). Its invariant polynomials \( u(\bar{z}_1, z_2) \) are precisely the polynomials in \( z_2, z_2 \bar{z}_1 + \bar{z}_1^2 \). The polynomial \( r(\bar{z}_1) \) can be uniquely written as
\[
r^+(z_2, \bar{z}_1^2 z_2 + \bar{z}_1^2) + (\bar{z}_1 + \frac{z_2}{2}) r^-(z_2, \bar{z}_1^2 z_2 + \bar{z}_1^2)
\] (160)
in two polynomials \( r^\pm \). Taking \( f = -r^- \) and \( g = r^+ - \frac{\bar{z}_1}{2} r^- \), we find the desired solutions. \( \square \)

14. Normal Form for General Type C.1 Submanifolds

In this section we show that generically a Levi-flat type C.1 submanifold is not formally equivalent to the quadric \( M_{C.1} \) submanifold. In fact, we find a formal normal form that shows infinitely many invariants. There are obviously infinitely many invariants if we do not impose the Levi-flat condition. The trick therefore is, how to impose the Levi-flat condition and still obtain a formal normal form.

Let \( M \subset \mathbb{C}^3 \) be a real-analytic Levi-flat type C.1 submanifold through the origin. We know that \( M \) is an image of \( \mathbb{R}^2 \times \mathbb{C} \) under a real-analytic CR map that is a diffeomorphism onto its target; see Theorem \[1.2\]. After a linear change of coordinates we assume that the mapping is
\[
(x, y, \xi) \in \mathbb{R}^2 \times \mathbb{C} \mapsto (x + iy + a(x, y, \xi), \xi + b(x, y, \xi), (x - iy)\xi + (x - iy)^2 + r(x, y, \xi)),
\] (161)
where \( a, b \) are \( O(2) \) and \( r \) is \( O(3) \). As the mapping is a CR mapping and a local diffeomorphism, then given any such \( a, b, \) and \( r \), the image is necessarily Levi-flat at CR points. Therefore the set of all these mappings gives us all type C.1 Levi-flat submanifolds.
We precompose with an automorphism of $\mathbb{R}^2 \times \mathbb{C}$ to make $b = 0$. We cannot similarly remove $a$ as any automorphism must have real valued first two components (the new $x$ and the new $y$), and hence those components can only depend on $x$ and $y$ and not on $\xi$. So if $a$ depends on $\xi$, we cannot remove it by precomposing.

Next we notice that we can treat $M$ as an abstract CR manifold. Suppose we have two equivalent submanifolds $M_1$ and $M_2$, with $F$ being the biholomorphic map taking $M_1$ to $M_2$. If $M_j$ is the image of a map $\varphi_j$, then note that $\varphi_2^{-1}$ is CR on $(M_2)_{CR}$. Therefore, $G = \varphi_2^{-1} \circ F \circ \varphi_1$ is CR on $(F \circ \varphi_1)^{-1}((M_2)_{CR})$, which is dense in a neighbourhood of the origin of $\mathbb{R}^2 \times \mathbb{C}$ (the CR singularity of $M_2$ is a thin set, and we pull it back by two real-analytic diffeomorphisms). A real-analytic diffeomorphism that is CR on a dense set is a CR mapping. The same argument works for the inverse of $G$, and therefore we have a CR diffeomorphism of $\mathbb{R}^2 \times \mathbb{C}$. The conclusion we make is the following proposition.

**Proposition 14.1.** If $M_j \subset \mathbb{C}^3$, $j = 1, 2$ are given by the maps $\varphi_j$

\[
(x, y, \xi) \in \mathbb{R}^2 \times \mathbb{C} \xrightarrow{\varphi_j} (x + iy + a_j(x, y, \xi), \\
\xi + b_j(x, y, \xi), \\
(x - iy)(\xi + (x - iy)^2 + r_j(x, y, \xi)),
\]

and $M_1$ and $M_2$ are locally biholomorphically (resp. formally) equivalent at 0, then there exists local biholomorphisms (resp. formal equivalences) $F$ and $G$ at 0, with $F(M_1) = M_2$, $G(\mathbb{R}^2 \times \mathbb{C}) = \mathbb{R}^2 \times \mathbb{C}$ as germs (resp. formally) and

\[
\varphi_2 = F \circ \varphi_1 \circ G.
\]

In other words, the proposition states that if we find a normal form for the mapping we find a normal form for the submanifolds. Let us prove that the proposition also works formally.

**Proof.** We have to prove that $G$ restricted to $\mathbb{R}^2 \times \mathbb{C}$ is CR, that is, $\frac{\partial G}{\partial \xi} = 0$. Let us consider

\[
\varphi_2 \circ G = F \circ \varphi_1.
\]

The right hand side does not depend on $\bar{\xi}$ and thus the left hand side does not either. Write $G = (G^1, G^2, G^3)$. Let us write $b = b_2$ and $r = r_2$ for simplicity. Taking derivative of $\varphi_2 \circ G$ with respect to $\xi$ we get:

\[
\begin{align*}
G^1_\xi + ig^2_\xi + a_x(G)G^1_\xi + a_y(G)G^2_\xi + a_\xi(G)G^3_\xi &= 0, \\
G^2_\xi + ib_x(G)G^1_\xi + b_y(G)G^2_\xi + b_\xi(G)G^3_\xi &= 0, \\
(G^1_\xi - ig^2_\xi)G^3 + (G^1 - ig^2)G^3_\xi + 2(G^1 - ig^2)(G^1_\xi - ig^2_\xi) + r_x(G)G^1_\xi + r_y(G)G^2_\xi + r_\xi(G)G^3_\xi &= 0.
\end{align*}
\]

Suppose that the homogeneous parts of $G^1_\xi$ are zero for all degrees up to degree $d - 1$. If we look at the degree $d$ homogeneous parts of the first two equations above we immediately note that it must be that $G^1_\xi + ig^2_\xi = 0$ and $G^3_\xi = 0$ in degree $d$. We then look at the degree $d + 1$ part of the third equation. Recall that $[\cdot]_d$ is the degree $d$ part of an expression. We get

\[
[G^1_\xi - ig^2_\xi]_d [G^3 + 2G^1 - i2G^2]_1 = 0.
\]
As $G$ is an automorphism we cannot have the linear terms be linearly dependent and hence $G^1_\xi = G^2_\xi = 0$ in degree $d$. We finish by induction on $d$. □

Using the proposition we can restate the result of Theorem 1.3 using the parametrization.

**Corollary 14.2.** A real-analytic Levi-flat type C.1 submanifold $M \subset \mathbb{C}^3$ is biholomorphically equivalent to the quadric $M_{C.1}$ if and only if the mapping giving $M$ is equivalent to a mapping of the form

$$(x, y, \xi) \in \mathbb{R}^2 \times \mathbb{C} \mapsto (x + iy, \xi, (x - iy)\xi + (x - iy)^2 + r(x, y, \xi)).$$

(167)

That is, $M$ is equivalent to $M_{C.1}$ if and only if we can get rid of the $a(x, y, \xi)$ via pre and post composing with automorphisms. The proof of the corollary follows as a submanifold that is realized by this map must be of the form $w = \bar{z}_1z_2 + \bar{z}_1^2 + \rho(z_1, \bar{z}_1, z_2)$ and we apply Theorem 1.3.

We have seen that the involution $\tau$ on $M$, in particular when $M$ is the quadric, is useful to compute the automorphism group and to construct Levi-flat submanifolds of type C.1. We will also need to deal with power series in $z, \bar{z}, \xi$. Thus we extend $\tau$, which is originally defined on $\mathbb{C}^3$, as follows

$$\sigma(z, \bar{z}, \xi) = (z, -\bar{z} - \xi, \xi).$$

(168)

Here $z, \bar{z}, \xi$ are treated as independent variables. Note that $z, \xi, w = \bar{z}\xi + \bar{z}^2$ are invariant by $\sigma$, while $\eta = \bar{z} + \frac{1}{2}\xi$ is skew invariant by $\sigma$. A power series in $z, \bar{z}, \xi$ that is invariant by $\sigma$ is precisely a power series in $z, \xi, w$. In general, a power series $u$ in $z, \bar{z}, \xi$ admits a unique decomposition

$$u(z, \bar{z}, \xi) = u^+(z, \xi, w) + \eta u^-(z, \xi, w).$$

(169)

First we introduce degree for power series $u(z, \bar{z}, \xi)$ and weights for power series $v(z, \xi, w)$. As usual we assign degree $i + j + k$ to the monomial $z^i\bar{z}^j\xi^k$. We assign weight $i + j + 2k$ to the monomial $z^i\bar{z}^j\xi^k$. For simplicity, we will call them weight in both situations. Let us also denote

$$[u]_d(z, \bar{z}, \xi) = \sum_{i+j+k=d} u_{ijk}z^i\bar{z}^j\xi^k, \quad [v]_d(z, \xi, w) = \sum_{i+j+2k=d} v_{ijk}z^i\bar{z}^j\xi^kw^k.$$  

(170)

Set $[u]^d_i = [u]_i + \cdots + [u]_j$ and $[v]^d_i = [v]_i + \cdots + [v]_j$ for $i \leq j$.

**Theorem 14.3.** Let $M$ be a real-analytic Levi-flat type C.1 submanifold in $\mathbb{C}^3$. There exists a formal biholomorphic map transforming $M$ into the image of

$$\hat{\varphi}(z, \bar{z}, \xi) = (z + A(z, \xi, w)w\eta, \xi, w)$$

(171)

with $\eta = \bar{z} + \frac{1}{2}\xi$ and $w = \bar{z}\xi + \bar{z}^2$. Suppose further that $A \neq 0$. Fix $i_s, j_s, k_s$ such that $j_s$ is the largest integer satisfying $A_{i, j, k} \neq 0$ and $i_s + j_s + 2k_s = s$. Then we can achieve

$$A_{i_s(j_s + n)k_s} = 0, \quad n = 1, 2, \ldots.$$  

(172)

Furthermore, the power series $A$ is uniquely determined up to the transformation

$$A(z, \xi, w) \mapsto c^3A(cz, \bar{c}\xi, c^2w), \quad c \in \mathbb{C} \setminus \{0\}.$$  

(173)

In the above normal form with $A \neq 0$, the group of formal biholomorphisms that preserve the normal form consists of dilations

$$(z, \xi, w) \mapsto (\nu z, \bar{\nu}\xi, \nu^2w)$$

(174)

satisfying $\bar{\nu}^3A(\nu z, \bar{\nu}\xi, \nu^2w) = A(z, \xi, w)$. 


Proof. It will be convenient to write the CR diffeomorphism $G$ of $\mathbb{R}^2 \times \mathbb{C}$ as $(G_1, G_2)$ where $G_1$ is complex-valued and depends on $z, \bar{z}$, while $G_2$ depends on $z, \bar{z}, \xi$. Let $M$ be the image of a mapping $\varphi$ defined by

$$
(z, \bar{z}, \xi) \mapsto \left( z + a(z, \bar{z}, \xi), \xi, \bar{z}\xi + \bar{z}^2 + r(z, \bar{z}, \xi) \right)
$$

with $a = O(2), r = O(3)$. We want to find a formal biholomorphic map $F$ of $\mathbb{C}^3$ and a formal CR diffeomorphism $G$ of $\mathbb{R}^2 \times \mathbb{C}$ such that

$$
F \hat{\varphi} G^{-1} = \varphi
$$

is in the normal form.

To simplify the computation, we will first achieve a preliminary normal form where $r = 0$ and the function $a$ is skew-invariant by $\sigma$. For the preliminary normal form we will only apply $F, G$ that are tangent to the identity. We will then use the general $F, G$ to obtain the final normal form.

Let us assume that $F, G$ are tangent to the identity. Let $M = F(\hat{\varphi}(\mathbb{R}^2 \times \mathbb{C}))$ where $\hat{\varphi}$ is determined by $\hat{a}, \hat{r}$. We write

$$
F = I + (f_1, f_2, f_3), \quad G = I + (g_1, g_2).
$$

The $\xi$ components in $\varphi G = F \hat{\varphi}$ give us

$$
g_2(z, \bar{z}, \xi) = f_2(z + \hat{a}(z, \bar{z}, \xi), \xi, \bar{z}\xi + \bar{z}^2 + \hat{r}(z, \bar{z}, \xi)).
$$

Thus, we are allowed to define $g_2$ by the above identity for any choice of $f_2 = O(2)$. Eliminating $g_2$ in other components of $\varphi G = F \hat{\varphi}$, we obtain

$$
f_1 \circ \hat{\varphi} - g_1 = a \circ G - \hat{a},
$$

$$
f_3 \circ \hat{\varphi} - \bar{z} f_2 \circ \hat{\varphi} = r \circ G - \hat{r} + (2\eta + \xi) g_1 + \bar{g}_1 f_2 \circ \hat{\varphi} + \bar{g}_1^2,
$$

where $\bar{g}_1(z, \bar{z}) = \bar{g}_1(\bar{z}, z)$ and

$$
(a \circ G)(z, \bar{z}, \xi) := a(G_1(z, \bar{z}), \bar{G}_1(z, \bar{z}), G_2(z, \bar{z}, \xi)).
$$

Each power series $r(z, \bar{z}, \xi)$ admits a unique decomposition

$$
r(z, \bar{z}, \xi) = r^+(z, \xi, w) + \eta r^-(z, \xi, w),
$$

where both $r^\pm$ are invariant by $\sigma$. Note that $r(z, \bar{z}, \xi)$ is a power series in $z, \xi$ and $w$, if and only if it is invariant by $\sigma$, i.e. if $r^- = 0$. We write

$$
r^+ = wt(k), \quad or \quad wt(r^+) \geq k,
$$

if $r^+ = 0$ for $a + b + 2c < k$. Define $r^- = wt(k)$ analogously and write $\eta r^- = wt(k)$ if $r^- = wt(k-1)$. We write $r = wt(k)$ if $(r^+, \eta r^-) = wt(k)$. Note that

$$
r = O(k) \Rightarrow r = wt(k); \quad wt(rs) \geq wt(r) + wt(s).
$$

The power series in $z, \bar{z}$ play a special role in describing normal forms. Let us define $T^\pm$ via

$$
u(z, \bar{z}) = (T^+ u)(z, \xi, w) + (T^- u)(z, \xi, w)\eta.
$$
Let $S_k^+$ (resp. $S_k^-$) be spanned by monomials in $z, \bar{z}, \xi$ which have weight $k$ and are invariant (resp. skew-invariant) by $\sigma$. Then the range of $\eta T^-$ in $S_k^-$ is a linear subspace $R_k$. We decompose
\[ S_k^- = R_k \oplus (S_k^- \ominus R_k). \]  
(186)

The decomposition is of course not unique. We will take
\[ S_k^- \ominus R_k = \bigoplus_{a+b+2c=k-1, c>0} \mathbb{C} z^a \xi^b w^c \eta. \]  
(187)

Here we have used $\eta = \bar{z} + \frac{1}{2} \xi$, $\eta^2 = w + \frac{1}{2} \xi^2$, and
\[ T^+ u(z, \xi, w) = \sum_{i,j \geq 0, 0 \leq \alpha \leq j/2} u_{ij} \left( \frac{j}{2\alpha} \right) z^i (w + \frac{1}{4} \xi^2)^\alpha (-\frac{1}{2} \xi)^{-2\alpha}. \]  
(188)

\[ T^- u(z, \xi, w) = \sum_{i \geq 0, j \geq 0, 0 \leq \alpha < j/2} u_{ij} \left( \frac{j}{2\alpha + 1} \right) z^i (w + \frac{1}{4} \xi^2)^\alpha (-\frac{1}{2} \xi)^{2\alpha-1}. \]  
(189)

In particular, we have
\[ T^- u(z, \xi, 0) = \sum_{i \geq 0, j > 0} (-1)^{j-1} u_{ij} z^i \xi^{j-1}. \]  
(190)

This shows that
\[ T^- u(z, \xi, 0) = \frac{1}{-\xi} \left( u(z, -\xi) - u(z, 0) \right). \]  
(191)

We are ready to show that under the condition that $g_1(z, \bar{z})$ has no pure holomorphic terms, there exists a unique $(F, G)$ which is tangent to the identity such that \( \hat{r} = 0 \) and
\[ \hat{a} \in \mathcal{N} := \bigoplus \mathcal{N}_k, \quad \mathcal{N}_k := S_k^- \ominus R_k. \]  
(192)

We start with terms of weight 2 in (179)-(180) to get
\[ [f_1]_2 - [g_1]_2 = [a]_2 - \eta [\hat{a}^-]_1, \]  
(193)
\[ [f_3]_2 = 0. \]  
(194)

Note that $f_j^- = 0$. The first identity implies that
\[ [f_1]_2 - [T^+ g_1]_2 = [a^+]_2, \quad [T^- g_1]_1 = [\hat{a}^-]_1 - [a^-]_1. \]  
(195)

The first equation is solvable with kernel defined by
\[ [f_1]_k - [T^+ g_1]_k = 0. \]  
(196)

for $k = 2$. This shows that $[g_1]_2$ is still arbitrary and we use it to achieve
\[ \eta [\hat{a}^-]_1 \in S_2^- \ominus R_2 = \{0\}. \]  
(197)

Then the kernel space is defined by (196) and
\[ [g_1(z, \bar{z}) - g_1(z, 0)]_k = 0 \]  
(198)

with $k = 2$. In particular, under the restriction
\[ [g_1(z, 0)]_k = 0, \]  
(199)

for $k = 2$, we have achieved $\hat{a}^- \in \mathcal{N}_2$ by unique $[f_1]_2, [g_1]_2, [f_2]_1, [f_3]_2$. By induction, we verify that if (199) holds for all $k$, we determine uniquely $[f_1]_k, [g_1]_k$ by normalizing $[\hat{a}]_k \in \mathcal{N}_k$. We
then determine \([f_2]_{k+1}, [f_3]_{k+1}\) uniquely to normalize \([\hat{r}]_{k+1} = 0\). For the details, let us find formula for the solutions. We rewrite (179) as

\[
T^-g_1 = -(a \circ G - \hat{a} - f_1 \circ \hat{\varphi})^-, \quad (f_1 \circ \hat{\varphi})^+ = (a \circ G - \hat{a})^+ + T^+g_1. \tag{200}
\]

Using (190), we can solve

\[
(-1)^{j-1}g_{1,ij} = -((a \circ G)^-)_{i(j-1)0}, \quad j \geq 1, \quad i+j = k. \tag{202}
\]

Then we have

\[
(\hat{a}^-)_{ij0} = 0, \quad i+j = k-1;
(\hat{a}^-)_{ijm} = ((a \circ G - f_1 \circ \hat{\varphi} + g_1^-)_{ijm}, \quad m \geq 1, i+j+m = k-1. \tag{204}
\]

Note that \(-[g_1]_k(z, -\bar{z}) = \bar{z}[(a \circ G - \hat{a})^-]_{k-1}(z, \bar{z}, 0)\). We obtain

\[
[g_1]_k(z, \bar{z}) = \bar{z}[(a \circ G - \hat{a})^-]_{k-1}(z, -\bar{z}, 0). \tag{205}
\]

Having determined \([g_1]_k\), we take

\[
[f_1]_k = [(a \circ G - \hat{a} + g_1)^+]_k. \tag{206}
\]

We then solve (180) by taking

\[
[f_2]_k = [E^-]_k, \quad [f_3]_{k+1} = [(E - \frac{1}{2}\xi f_2^+)_{k+1}, \quad E := r \circ G - \hat{r} + (2\eta + \xi)\hat{g}_1 + \hat{g}_1 f_2 \circ \hat{\varphi} + \hat{g}_1^2. \tag{208}
\]

We have achieved the preliminary normalization.

Assume now that

\[
\varphi(z, \bar{z}, \xi) = (z + a^-(z, \xi, w)\eta, \xi, w), \quad \hat{\varphi}(z, \bar{z}, \xi) = (z + \hat{a}^- (z, \xi, w)\eta, \xi, w) \tag{209}
\]

are in the preliminary normal form, i.e.

\[
w|a^-(z, \xi, w), \quad w|\hat{a}^-(z, \xi, w). \tag{210}
\]

Let us assume that

\[
a^-(z, \xi, w) = wt(s), \quad [a^-]_s \neq 0; \quad \hat{a}^-(z, \xi, w) = wt(s). \tag{211}
\]

We assume that \(\varphi G = F\hat{\varphi}\) with

\[
F(z, \xi, w) = I + (f_1, f_2, f_3), \quad G(z, \bar{z}, \xi) = (z + g_1(z, \bar{z}), \xi + g_2(z, \bar{z}, \xi)). \tag{213}
\]

Here \(f_i, g_j\) start with terms of weight and order at least 2. In particular, we have

\[
f_i = wt(N), \quad g_i = wt(N), \quad i = 1, 2; \quad f_3 = wt(N'); \quad N' \geq N \geq 2. \tag{214}
\]
Set $(P, Q, R) := \varphi G$. Using $N \geq 2$, $s \geq 2$, and the Taylor theorem, we obtain

\[ P = z + g_1(z, \bar{z}) + a^-(z, \xi, w)\eta + a^-(z, \xi, w)(\bar{g}_1(\bar{z}, z) + \frac{1}{2}g_2(z, \bar{z}, \xi)) \]

\[ + \eta \nabla a^- (z, \xi, w) \cdot \left(g_1(z, \bar{z}), g_2(z, \bar{z}, \xi), (\xi + 2\bar{z})\bar{g}_1(\bar{z}, z) + \bar{z}g_2(z, \bar{z}, \xi)\right) \]

\[ + wt(s + N + 1), \]

\[ Q = \xi + g_2(z, \bar{z}, \xi), \]

\[ R = w + (2\bar{z} + \xi)\bar{g}_1(\bar{z}, z) + \bar{z}g_2(z, \bar{z}, \xi) + wt(2N). \]

We also have $(P, Q, R) = F\dot{\varphi}$. Thus

\[ P = z + \hat{a}^- (z, \xi, w)\eta + f_1(z, \xi, w) + \partial_z f_1(z, \xi, w)\hat{a}^- (z, \xi, w)\eta + wt(N + s + 1), \]

\[ Q = \xi + f_2(z, \xi, w) + \partial_z f_2(z, \xi, w)\hat{a}^- (z, \xi, w)\eta + wt(N + s + 1), \]

\[ R = w + f_3(z, \xi, w) + \partial_z f_3(z, \xi, w)\hat{a}^- (z, \xi, w)\eta + wt(N' + s + 1). \]

We will use the above 6 identities for $P, Q, R$ in two ways. First we use their lower order terms to get

\[ f_1(z, \xi, w) = g_1(z, \bar{z}) + (a^- (z, \xi, w) - \hat{a}^- (z, \xi, w))\eta + wt(N + s), \]

\[ f_2(z, \xi, w) = g_2(z, \bar{z}, \xi) + wt(N + s), \]

\[ f_3(z, \xi, w) = (2\bar{z} + \xi)\bar{g}_1(\bar{z}, z) + \bar{z}g_2(z, \bar{z}, \xi) + wt(2N) + wt(N' + s). \]

Hence, we can take $N' = N + 1$. By \[223\] and the preliminary normalization, we first know that

\[ \hat{a} = a + wt(N + s - 1), \]

\[ f_1(z, \xi, w) = b(z) + wt(N + s), \quad g_1(z, \bar{z}) = b(z) + wt(N + s). \]

We compose \[225\] by $s$ and then take the difference of the two equations to get

\[ f_2(z, \xi, w) = -\bar{b}(\bar{z}) - \bar{b}(-\bar{z} - \xi) + wt(2N - 1) + wt(N + s), \]

\[ f_3(z, \xi, w) = -\bar{\bar{z}}\bar{b}(-\bar{z} - \xi) + (\bar{z} + \xi)\bar{b}(\bar{z}) + wt(2N) + wt(N + s + 1). \]

Here we have used $N' = N + 1$. Let $b(z) = b_N z^N + wt(N + 1)$. Therefore, we have

\[ g_2(z, \bar{z}, \xi) = -\bar{b}_N(\bar{z}^N + (-\bar{z} - \xi)^N) + wt(N + 1), \]

\[ \bar{g}_1(\bar{z}, z) + \frac{1}{2}g_2(z, \bar{z}, \xi) = \eta \bar{b}_N \sum_i \bar{z}^i (-\bar{z} - \xi)^{N-1-i} + wt(N + 1), \]

\[ (2\bar{z} + \xi)\bar{g}_1(\bar{z}, z) + \bar{z}g_2(z, \bar{z}, \xi) = \bar{b}_N(\bar{z}^{N-1} + (-\bar{z} - \xi)^{N-1})w + wt(N + 2). \]

Next, we use the two formulae for $P$ and \[227\] to get the identity in higher weight:

\[ \hat{a}^- = a^- + g_1^- + Lb_N + wt(N + s), \quad f_1^- = g_1^- + wt(N + s + 1). \]

Here we have used $f_1^- = 0$ and

\[ Lb_N(z, \xi, w) := -N b_N z^{N-1}a^- s(z, \xi, w) - [a^- s(z, \xi, w)] \bar{b}_N \sum_i \bar{z}^i (-\bar{z} - \xi)^{N-1-i} \]

\[ + \nabla [a^- s] \cdot \left(b_N z^N, -\bar{b}_N(\bar{z}^N + (-\bar{z} - \xi)^N), \bar{b}_N w(\bar{z}^{N-1} + (-\bar{z} - \xi)^{N-1})\right). \]
Recall that $w|a^-$ and $w|a^-$. We also have that $w|Lb_N(z, \xi, w)$ and $Lb_N$ is homogenous in weighted variables and of weight $N + s - 1$. This shows that $[g_1^-(z, \xi, 0)]_{N+s-1} = 0$. By (190), we get

$$[g_1(z, \bar{z})]_{N+s} = [g_1(z, 0)]_{N+s}, \quad [\hat{a}^-]_{s+N-1} = [a^-]_{s+N-1} + Lb_N. \quad (235)$$

Let us make some observations. First, $Lb_N$ depends only on $b_N$ and it does not depend on coefficients of $b(z)$ of degree larger than $N$. We observe that the first identity says that all coefficients of $[g_1]_{N+s}$ must be zero, except that the coefficient $g_1(N+s)0$ is arbitrary. On the other hand $Lb_N$, which has weight $N + s - 1$, depends only on $g_1,N0$, while $N + s - 1 > N$. Let us assume for the moment that we have $Lb_N \neq 0$ for all $b_N \neq 0$. We will then choose a suitable complement subspace $N_{N+s-1}$ of the space of weighted homogenous polynomials in $z, \xi, w$ of weight $N + s - 1$ for $Lb_N$. Then $\hat{a}^- \in w\sum_{s=N+1}N_{N+s-1}$ will be the required normal form. The normal form will be obtained by the following procedures: Assume that $\varphi$ is not formally equivalent to the quadratic mapping in the preliminary normalization. We first achieve the preliminary normal form by a mapping $F^0 = I + (f_1^0, f_2^0, f_3^0)$ and $G^0 = I + (g_1^0, g_2^0)$ which are tangent to the identity. We can make $F^0, G^0$ to be unique by requiring $\sum_{s=1}^3 f_i(z, 0) = 0$. Then $a$ is normalized such that $\hat{a}^- = \hat{a}^- \eta$ with $[\hat{a}^-]_s$ being non-zero homogenous part of the lowest weight. We may assume that $[a]_{s+1} = [\hat{a}]_{s+1}$. Inductively, we choose $f_{1,N0}^i (N = 2, 3, \ldots)$ to achieve $[\hat{a}^-]_{N+s-1} \in \hat{w}N_{N+s-1}$. In this step for a given $N$, we determine mappings $F^1 = I + (f_1^1, f_2^1, f_3^1)$ and $G^1 = I + (g_1^1, g_2^1)$ by requiring that $f_i^1(z, \xi, w)$ contains only one term $\xi^N$, while $f_1^1, f_2^1, g_1^1, g_2^1$ have weight at most $N$ and $f_3^1$ has weight at most $N + 1$. In the process, we also show that $[f_i^1(z, \xi, w)]_{N+s}^1$ depends only on $z$, if we do not want to impose the restriction on $f_i^1$. Moreover, the coefficient of $\xi^{N+s-1}$ of $f_i^1$ can still be arbitrarily chosen without changing the normalization achieved for $[\hat{a}^-]_{N+s-1}$ via $[f_1^1]_N$. However, by achieving $[\hat{a}^-]_{N+s-1} \in \hat{w}N_{N+s-1}$ via $F^1, G^1$, we may destroy the preliminary normalization achieved via $F_0, G_0$. We will then restore the preliminary normalization via $F^2 = I + (f_1^2, f_2^2, f_3^2), G^2 = I + (g_1^2, g_2^2)$ satisfying $g_i^2(z, 0) = 0$. This amounts to determining $g_2^2 = g_1$ and $f_1^2 = f_1$ via (200) and (201) for which the terms of weight at most $N + s$ have been determined by (233), and then $f_2^2 = f_2, f_3^2 = f_3, g_2^2 = g_2$ are determined by (207)-(208) and (178), respectively. This allows us to repeat the procedure to achieve the normalization in any higher weight. We will then remove the restriction that the normalizing mappings must be tangent to the identity. This will alter the normal form only by suitable linear dilations.

Suppose that $b_N \neq 0$. Let us verify that

$$Lb_N \neq 0. \quad (236)$$

We will also identify one of non-zero coefficients to describe the normalizing condition on $\hat{a}$. We write the two invariant polynomials

$$z^N + (-\bar{z} - \xi)^N = \lambda_N \xi^N + \sum_{j< N} p_{ijk} z^i \xi^j w^k, \quad (237)$$

$$\sum_i z^i (-\bar{z} - \xi)^{N-1-i} = \lambda_{N-1} \xi^{N-1} + \sum_{j<N-1} q_{ijk} z^i \xi^j w^k. \quad (238)$$
If we plug in $w = \bar{z}^2 + z\xi$ we obtain a polynomial identity in the variables $z, \bar{z}, \xi$.

$$
\bar{z}^N + (-\bar{z} - \xi)^N = \lambda_N z^N + \sum_{j \leq N} p_{ijk} z^j \xi^i (\bar{z}^2 + z\xi)^k, \tag{239}
$$

$$
\sum_i \bar{z}^i (-\bar{z} - \xi)^{N-1-i} = \lambda'_{N-1} z^{N-1} + \sum_{j \leq N-1} q_{ijk} z^j \xi^i (\bar{z}^2 + z\xi)^k. \tag{240}
$$

If we set $\bar{z} = z = 0$, we obtain that

$$
\lambda_N = \lambda'_{N} = (-1)^N. \tag{241}
$$

Recall that $j_s$ is the largest integer such that $(a^{-})_{i_s,j_s,k_s} \neq 0$ and $i_s + j_s + 2k_s = s$. Since $w|[a^{-}]_{s}$, then $k_s > 0$. We obtain

$$
(Lb_{N})_{i_s(j_s+N-1)k_s} = (a^{-})_{i_s,j_s,k_s} b_{N} (-\lambda'_{N-1} - j_s \lambda_{N-1} + k_s \lambda_{N}) \neq 0. \tag{242}
$$

Therefore, we can achieve

$$
(\hat{a}^{-})_{i_s(j_s+n)k_s} = 0, \quad n = 1, 2, \ldots. \tag{243}
$$

This determines uniquely all $b_2, b_3, \ldots$.

We now remove the restriction that $F$ and $G$ are tangent to the identity. Suppose that both $\varphi$ and $\hat{\varphi}$ are in the normal form. Suppose that $F\varphi = \hat{\varphi}G$. Then looking at the quadratic terms, we know that the linear parts $F, G$ must be dilations. In fact, the linear part of $F$ must be the linear automorphism of the quadric. Thus the linear parts of $F$ and $G$ have the forms

$$
G': (z, \xi) = (\nu z, \nu \xi), \quad F'(z, \xi, w) = (\nu z, \nu \xi, \nu^2 w). \tag{244}
$$

Then $(F')^{-1} \hat{\varphi} G'$ is still in the normal form. Since $(F')^{-1} F$ is holomorphic and $(G')^{-1} G$ is CR, by the uniqueness of the normalization, we know that $F' = F$ and $G' = G$. Therefore, $F$ and $G$ change the normal form $a^{-}$ as follows

$$
a^{-}(z, \xi, w) = \nu \hat{a}^{-}(\nu z, \nu \xi, \nu^2 w), \quad \nu \in \mathbb{C} \setminus \{0\}. \tag{245}
$$

When $[\hat{a}^{-}]_{s} = [a^{-}]_{s} \neq 0$, we see that $|\nu| = 1$. Therefore, the formal automorphism group is discrete or one-dimensional. \hfill \square

In [3], Coffman used an analogous method of even/odd function decomposition to study a quadratic normal form for non Levi-flat real analytic 4-submanifolds in $\mathbb{C}^3$. He was able to achieve the convergent normalization by a rapid iteration method. Using the above decomposition of invariant and skew-invariant functions of the involution $\sigma$, one might achieve a convergent solution for approximate equations when $M$ is formally equivalent to the quadric. However, when the iteration is employed, each new CR mapping $\hat{\varphi}$ might only be defined on a domain that is proportional to that of the previous $\varphi$ in a constant factor. This is significantly different from the situations of Moser [28] and Coffman [8, 10], where rapid iteration methods are applicable. Therefore, even if $M$ is formally equivalent to the quadric, we do not know if they are holomorphically equivalent.

15. Instability of Bishop-like submanifolds

Let us now discuss stability of Levi-flat submanifolds under small perturbations that keep the submanifolds Levi-flat, in particular we discuss which quadratic invariants are stable when moving from point to point on the submanifold. The only stable submanifolds are A$n$
and C.1. The Bishop-like submanifolds (or even just the Bishop invariant) are not stable under perturbation, which we show by constructing examples.

**Proposition 15.1.** Suppose that \( M \subset \mathbb{C}^{n+1}, \ n \geq 2, \) is a connected real-analytic real codimension 2 submanifold that has a non-degenerate CR singular at the origin. \( M \) can be written in coordinates \((z,w) \in \mathbb{C}^n \times \mathbb{C} \) as

\[
w = A(z, \bar{z}) + B(\bar{z}, z) + O(3),
\]

for quadratic \( A \) and \( B. \) In a neighborhood of the origin all complex tangents of \( M \) are non-degenerate, while ranks of \( A, B \) are upper semicontinuous. Suppose that \( M \) is Levi-flat (that is \( M_{CR} \) is Levi-flat). The CR singular set of \( M \) that is not of type \( B_{1/2} \) at the origin is a real analytic subset of \( M \) of codimension at least 2, while the CR singular set of \( M \) that is of type \( B_{1/2} \) the origin has codimension at least 1. \( A_n \) has an isolated CR singular point at the origin and so does C.1 in \( \mathbb{C}^3. \) Let \( S_0 \subset M \) be the set of CR singular points. There is a neighborhood \( U \) of the origin such that for \( S = S_0 \cap U \) we have the following.

(i) If \( M \) is of type \( A.k \) for \( k \geq 2 \) at the origin, then it is of type \( A.j \) at each point of \( S \) for some \( j \geq k. \)

(ii) If \( M \) is of type \( C.1 \) at the origin, then it is of type \( C.1 \) on \( S. \) If \( M \) is of type \( C.0 \) at the origin, then it is of type \( C.0 \) or \( C.1 \) on \( S. \)

(iii) There exists an \( M \) that is of type \( B.\gamma \) at one point and of \( C.1 \) at \( CR \) singular points arbitrarily near. Similarly there exists an \( M \) of type \( A.1 \) at \( p \in M \) that is either of type \( C.1, \) or \( B.\gamma, \) at points arbitrarily near \( p. \) There also exists an \( M \) of type \( B.\gamma \) at every point but where \( \gamma \) varies from point to point.

**Proof.** First we show that the rank of \( A \) and the rank of \( B \) are lower semicontinuous on \( S, \) without imposing Levi-flatness condition. Similarly the real dimension of the range of \( A(z, \bar{z}) \) is lower semicontinuous on \( S. \) Write \( M \) as

\[
w = \rho(z, \bar{z}),
\]

where \( \rho \) vanishes to second order at 0. If we move to a different point of \( S \) via an affine map \((z,w) \mapsto (Z + z_0, W + w_0). \) Then we have

\[
W + w_0 = \rho(Z + z_0, \bar{Z} + z_0).
\]

We compute the Taylor coefficients

\[
W = \frac{\partial \rho}{\partial z}(z_0, \bar{z}_0) \cdot Z + \frac{\partial \rho}{\partial \bar{z}}(z_0, \bar{z}_0) \cdot \bar{Z} +
\]

\[
+ Z^* \left[ \frac{\partial^2 \rho}{\partial z \partial \bar{z}}(z_0, \bar{z}_0) \right] Z + \frac{1}{2} Z^t \left[ \frac{\partial^2 \rho}{\partial z \partial \bar{z}}(z_0, \bar{z}_0) \right] Z + \frac{1}{2} \bar{Z}^t \left[ \frac{\partial^2 \rho}{\partial \bar{z} \partial \bar{z}}(z_0, \bar{z}_0) \right] \bar{Z} + O(3).
\]

The holomorphic terms can be absorbed into \( W. \) If \( \frac{\partial \rho}{\partial \bar{z}}(z_0, \bar{z}_0) \cdot \bar{Z} \) is nonzero, then this complex defining function has a linear term in \( W \) and linear term in \( \bar{Z} \) and the submanifold is CR at this point. Therefore the set of complex tangents of \( M \) is defined by

\[
\frac{\partial \rho}{\partial \bar{z}} = 0
\]

and each complex tangent point is non-degenerate. At a complex tangent point at the origin, \( A \) is given by \( \left[ \frac{\partial^2 \rho}{\partial z \partial \bar{z}}(z_0, \bar{z}_0) \right] \) and \( B \) is given by \( \frac{1}{2} \left[ \frac{\partial^2 \rho}{\partial \bar{z} \partial \bar{z}}(z_0, \bar{z}_0) \right]. \) In particular these matrices change continuously as we move along \( S. \) We first conclude that all CR singular points of \( M \)

in a neighborhood of the origin are non-degenerate. Further holomorphic transformations act on $A$ and $B$ using Proposition 2.1. Therefore the ranks of $A$ and $B$ as well as the real dimension of the range of $A(z, \bar{z})$ are lower semicontinuous on $S$ as claimed. Furthermore as $M$ is real-analytic, the points where the rank drops lie on a real-analytic subvariety of $S$, or in other words a thin set.

Imposing the condition that $M$ is Levi-flat, we apply Theorem 1.1. By a simple computation, the set of complex tangents of $M$ has codimension at least 2; and $A_\alpha$ has isolated CR singular point and so does $C.1$ in $\mathbb{C}^3$. The item (i) follows as A.$k$ are the only types where the rank of $B$ is greater than 1, and the theorem says $M$ must be one of these types. For (ii) note that since $A$ is of rank 1 when $M$ as $C.x$ at a point, $M$ cannot be of type A.$k$ nearby. If $M$ is of type C.1 at a point then the range of $A$ must be of real dimension 2 in a neighbourhood, and hence on this neighbourhood $M$ cannot be of type B.$\gamma$.

The examples proving item (iii) are given below.

**Example 15.2.** Define $M$ via

$$w = |z_1|^2 + \gamma \bar{z}_1^2 + \bar{z}_1 z_2 z_3.$$ (251)

It is Levi-flat by Proposition 6.2. At the origin $M$ is a type B.$\gamma$, but at a point where $z_1 = z_2 = 0$ but $z_3 \neq 0$, the submanifold is CR singular and it is of type C.1.

**Example 15.3.** Similarly if we define $M$ via

$$w = \bar{z}_1^2 + \bar{z}_1 z_2 z_3,$$ (252)

we obtain a CR singular Levi-flat $M$ that is A.1 at the origin, but C.1 at nearby CR singular points.

**Example 15.4.** If we define $M$ via

$$w = \gamma \bar{z}_1^2 + |z_1|^2 z_2,$$ (253)

then $M$ is a CR singular Levi-flat type A.1 submanifold at the origin, but type B.$\gamma$ at points where $z_1 = 0$ but $z_2 \neq 0$.

**Example 15.5.** The Bishop invariant can vary from point to point. Define $M$ via

$$w = |z_1|^2 + \bar{z}_1^2 (\gamma_1 (1 - z_2) + \gamma_2 z_2),$$ (254)

where $\gamma_1, \gamma_2 \geq 0$. It is not hard to see that $M$ is Levi-flat. Again it is an image of $\mathbb{C}^2 \times \mathbb{R}^2$ in a similar way as above.

At the origin, the submanifold is Bishop-like with Bishop invariant $\gamma_1$. When $z_1 = 0$ and $z_2 = 1$, the Bishop invariant is $\gamma_2$. In fact when $z_1 = 0$, the Bishop invariant at that point is

$$|\gamma_1 (1 - z_2) + \gamma_2 z_2|.$$ (255)

Proposition 6.2 says that this submanifold possesses a real-analytic foliation extending the Levi-foliation through the singular points. Proposition 6.1 says that if a foliation on $M$ extends to a (nonsingular) holomorphic foliation, then the submanifold would be a simple product of a Bishop submanifold and $\mathbb{C}$. Therefore, if $\gamma_1 \neq \gamma_2$ then the Levi-foliation on $M$ cannot extend to a holomorphic foliation of a neighbourhood of $M$. 

□
References


