1. Number Systems

1.1. The Real Numbers.

As in the text, we take an axiomatic approach to the properties of the real numbers. The set of real numbers is denoted by $\mathbb{R}$. We shall assume that the elements of $\mathbb{R}$ satisfy three kinds of axioms.

1.1.1. The Algebraic Axioms.

Given any two real numbers $x, y \in \mathbb{R}$, there is a uniquely determined real number $x + y \in \mathbb{R}$ called the sum. Given any two real numbers $x, y \in \mathbb{R}$, there is a uniquely determined real number $xy \in \mathbb{R}$ called the product. These two binary operations satisfy:

**Axiom 1** (Commutative Laws) For all $x, y \in \mathbb{R}$,

\[ x + y = y + x, \quad xy = yx. \]

**Axiom 2** (Associative Laws) For all $x, y, z \in \mathbb{R}$,

\[ (x + y) + z = x + (y + z), \quad (xy)z = x(yz). \]

**Axiom 3** (Distributive Law) For all $x, y, z \in \mathbb{R}$

\[ x(y + z) = xy + xz. \]

**Axiom 4** (Existence of Identities) There exist two distinct real numbers, denoted by 0 and 1, so that for all $x \in \mathbb{R}$

\[ x + 0 = 0 + x = x; \]
\[ x 1 = 1x = x. \]

**Axiom 5** (Existence of Negatives) For every real number $x \in \mathbb{R}$ there is a real number $y \in \mathbb{R}$ such that

\[ x + y = y + x = 0. \]

[One proves that there is only one such real number $y$, and it is denoted by $-x$.]

**Axiom 6** (Existence of Inverses) For every real number $x \neq 0$, there exists a real number $y \in \mathbb{R}$ such that

\[ xy = yx = 1. \]

[One proves that there is only one such real number $y$, and it is denoted by $\frac{1}{x}$.

1.1.2. The Order Axioms.

There exists a subset $\mathbb{R}^+ \subset \mathbb{R}$, called the set of positive real numbers, which satisfies the following axioms:

**Axiom 7** If $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$ then $x + y \in \mathbb{R}^+$ and $xy \in \mathbb{R}^+$.

**Axiom 8** For every real number $x \in \mathbb{R}$, exactly one of the following possibilities holds: either $x = 0$ or $x \in \mathbb{R}^+$ or $-x \in \mathbb{R}^+$. 
1.1.3. The Completeness Axiom.

There are many examples of sets satisfying these eight axioms. One familiar example is the subset $\mathbb{Q} \subset \mathbb{R}$ of real numbers which are **rational**; that is,

$$
\mathbb{Q} = \left\{ x \in \mathbb{R} \mid \text{there exist integers } p, q \in \mathbb{Z} \text{ with } q \neq 0 \text{ such that } x = \frac{p}{q} \right\}.
$$

Check that $\mathbb{Q}$ does satisfy Axiom 1 through Axiom 9. If this is the case, why don’t we just use rational numbers? At least part of the reason is that certain numbers which we often need are not rational.

**Theorem 1.** There is no rational number $x \in \mathbb{Q}$ such that $x^2 - 2 = 0$.

**Proof.** We argue by contradiction. Suppose that there were a rational number $x$ such that $x^2 = 2$. Then by definition, there exist integers $p$ and $q$ with $q \neq 0$ such that $x = \frac{p}{q}$. Note that we can assume that:

- At least one of $p$ or $q$ is odd

since otherwise we could cancel a factor of 2 from both $p$ and $q$ when we write $x = \frac{p}{q}$. Now since $x^2 = 2$, it follows that

$$
\left( \frac{p}{q} \right)^2 = 2
$$

and so

$$p^2 = 2q^2.
$$

But this implies that $p^2$ is even. Since the product of an odd number with itself is again odd, it follows that $p$ must be even, and so we can write

$$
p = 2s.
$$

Substituting in the equation above, we get

$$(2s)^2 = 2q^2$$

and hence

$$4s^2 = 2q^2.$$

Canceling a factor of 2 from both sides, we get

$$2s^2 = q^2.$$

But now this shows that $q^2$ is even, and by the same argument we used above in dealing with $p$, it follows that $q$ is even. Thus we can write

$$
q = 2t.
$$

Thus equations (1) and (2) show that both $p$ and $q$ are even, and this contradicts the hypothesis in the box above. This contradiction arose from the assumption that there was a rational number $x$ with $x^2 = 2$, and hence we can conclude that there is no such rational number. \[\square\]
We have shown that the numbers $\pm \sqrt{2}$ are not rational numbers. Since we shall want to use the number $\sqrt{2} = 1.414213562...$ as a real number, the theorem shows that the rational numbers $\mathbb{Q}$ are not ‘rich’ enough. It also shows that a system of numbers satisfying the first eight axioms does not necessarily contain all real numbers. What additional property do we need?

Suppose that $S \subset \mathbb{R}$ is a non-empty set of numbers. Then

- The set $S$ is **bounded above** if there is a number $M \in \mathbb{R}$ so that for every $x \in S$ it follows that $x \leq M$. In this case, $M$ is called an **upper bound** for $S$.

- A real number $\bar{M}$ is a **least upper bound** for the set $S$ if
  
  (1) $\bar{M}$ is an upper bound for $S$; that is, if $x \in S$ then $x \leq \bar{M}$.

  (2) If $M$ is any upper bound for $S$, then $\bar{M} \leq M$; in other words, there is no upper bound for $S$ which is strictly less than $\bar{M}$.

The crucial axiom which distinguishes the set of real numbers $\mathbb{R}$ from all other number systems satisfying Axioms 1 - 8 is then:

**Axiom 9** Every nonempty set $S \subset \mathbb{R}$ of real numbers which is bounded above has a least upper bound.

It may not be immediately clear why this assumption or Axiom is so important. However, we will see how powerful it is several times during the semester.

1.2. The Complex Numbers.

However even the real numbers are not completely satisfactory as a number system. For example, there is no real number $x$ such that $x^2 + 2 = 0$. (Prove this!) We fix this difficulty by introducing the set of complex numbers $\mathbb{C}$. To do this, we introduce a symbol $i$ (which will have the property that $i^2 = -1$). Then the set of complex numbers $\mathbb{C}$ is the set

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}.$$

Thus examples of complex numbers are $1 + 2i$, $-5 + 123i$, $\sqrt{2} - \frac{3}{2}i$, etc. We usually write the complex number $a + 0i$ simply as $a$, and this allows us to think of the real numbers as a subset of the complex numbers. Complex numbers of the form $0 + bi$ are written simply as $bi$, and such numbers are called **purely imaginary**. The number $0 + 0i = 0$ is just called **zero** and the number $1 + 0i = 1$ is just called **one**.

We can add complex numbers using the following rule:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$  (3)

We can also multiply complex numbers. If we multiply out formally and assume multiplication is commutative, we get

$$(a + bi)(c + di) = ac + a(di) + (bi)c + (bi)(di) = ac + adi + bci + bdi^2.$$  

Thus if we also assume $i^2 = -1$, we get the formula

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$  (4)
which is again a complex number. It is now relatively easy to check that the complex
numbers, with the definitions of addition and multiplication given in (3) and (4),
satisfy Axioms 1 through 5.

It turns out that they also satisfy Axiom 6: for every complex number \( a + bi \) which
is not \( 0 = 0 + 0i \) there is another complex number \( x + iy \) such that \( (a + bi)(x + iy) = 1 \).

To see this, we introduce the following definition:

If \( z = a + bi \) is a complex number, then the complex conjugate of \( z \), written \( \bar{z} \), is
the complex number

\[ \bar{z} = a - bi. \]

In other words, we get the complex conjugate of a complex number by changing
the sign of the imaginary part. But now for any complex number \( z = a + bi \), we
have

\[ z \bar{z} = (a + bi)(a - bi) = a^2 + b^2. \]  

(5)

Thus \( z \bar{z} \) is a real number, and in addition, this real number is strictly positive
unless \( a = b = 0 \).

Now suppose we want to divide one complex number \( z = a + bi \) by another
complex number \( w = x + iy \). Formally we have

\[
\frac{z}{w} = \frac{a + bi}{x + yi} = \frac{a + bi}{x + yi} \cdot 1
= \frac{a + bi}{x + yi} \cdot \frac{x - yi}{x - yi}
= \frac{(a + bi)(x - yi)}{(x + yi)(x - yi)}
= \frac{(ax + by) + (bx - ay)i}{x^2 + y^2}
= \left(\frac{ax + by}{x^2 + y^2}\right) + \left(\frac{bx - ay}{x^2 + y^2}\right)i.
\]

But this last expression is a complex number, and so we have shown that the
quotient of \( z \) by \( w \) is indeed a complex number. Note that what we did was to
multiply the numerator and denominator of \( \frac{z}{w} \) by \( \bar{w} \). In particular, to find the
inverse of the complex number \( w = x + yi \) we write

\[
\frac{1}{x + yi} = \frac{1}{x + yi} \cdot \frac{x - yi}{x - yi} = \frac{x - yi}{x^2 + y^2} = \left(\frac{x}{x^2 + y^2}\right) - \left(\frac{y}{x^2 + y^2}\right)i.
\]
1.3. Geometric Interpretation of \( \mathbb{C} \).

1.3.1. Cartesian coordinates.

We often think of the real numbers \( \mathbb{R} \) as corresponding to the points on a line. Using similar ideas, we can think of the complex numbers as corresponding to points in a plane. We usually take the horizontal axis to be the set of purely real numbers, while the vertical axis is the set of purely imaginary numbers. A complex number \( z = a + bi \) then corresponds to the point in the plane with Cartesian coordinates \( (a, b) \). Note that the complex conjugate \( \bar{z} = a - bi \) is obtained by reflecting the point \( z \) in the real axis. Using the formula for the distance between two points in the plane, it follows that the distance from a complex number \( z = a + bi \) to the origin \( 0 + 0i \) is \( \sqrt{a^2 + b^2} \). We call this the absolute value of the complex number \( z \), and write it as \( |z| \). This generalizes the notion of the absolute value of a real number. Note that we have

\[
|z|^2 = z\bar{z} = (a + bi)(a - bi) = a^2 + b^2
\]

\[
|z| = \sqrt{z\bar{z}} = (a^2 + b^2)^{\frac{1}{2}}.
\]

The absolute value of any complex number is a real number which is greater than or equal to zero. Moreover, the absolute value of a complex number is zero if and only if the complex number itself is zero.

**Exercise:** Prove that if \( z \) and \( w \) are complex numbers then

1. \( |zw| = |z||w|; \)
2. \( |z + w| \leq |z| + |w| \).

1.3.2. Polar coordinates.

There is another important way of thinking about complex numbers. If \( z = a + bi \) is a complex number, then we consider the line joining the point 0 and the point \( z \). This line has length \( |z| = \sqrt{a^2 + b^2} \). Also, this line makes an angle \( \theta \) with the positive real axis. If we extend this line (if necessary) so that it intersects the unit circle, the coordinates of the point of intersection are \((\cos(\theta), \sin(\theta))\), and thus corresponds to the complex number \( \cos(\theta) + \sin(\theta)i \). The complex number \( z \) is obtained by multiplying the complex number \( \cos(\theta) + \sin(\theta)i \) by the real number \( |z| \). Thus we have

\[
z = |z|(\cos(\theta) + \sin(\theta)i).
\]

One virtue of these coordinates is that it becomes much easier to multiply complex numbers. Thus suppose that \( z \) and \( w \) are two complex numbers, with \( z = |z|(\cos(\theta) + \sin(\theta)i) \) and \( w = |w|(\cos(\varphi) + \sin(\varphi)i) \). Then

\[
zw = |z||w|(\cos(\theta) + \sin(\theta)i)(\cos(\varphi) + \sin(\varphi)i)
\]

\[
= |z||w| \left( \cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi)i \right)
\]

\[
= |z||w| \left( \cos(\theta + \varphi) + \sin(\theta + \varphi)i \right).
\]

Thus to multiply two complex numbers, we only need to multiply the (real) absolute values and add the angles.
1.4. Finding roots.

We introduced complex numbers in order to find square roots of negative real numbers. It turns out that complex numbers in fact provide the solution to a more general problem: finding $N^{th}$-roots. If $z$ is any complex number, we say that a complex number $w$ is an $N^{th}$-root of $z$ if $w^N = z$. We can now show that every non-zero complex number has precisely $N$ $N^{th}$-roots.

Let us begin with an example: find the square roots of $i$. The trick is to write $i$ in terms of its polar coordinates. Since $|i| = 1$, and the line joining 0 to $i$ makes an angle of $\frac{\pi}{2}$ radians with the positive real axis, we can write

$$i = 1 \left( \cos \left( \frac{\pi}{2} \right) + \sin \left( \frac{\pi}{2} \right) i \right).$$

We now want to find complex numbers $z$ such that $z^2 = i$. Let us write

$$z = r \left( \cos(\theta) + \sin(\theta) i \right)$$

Then

$$z^2 = r^2 \left( \cos(2\theta) + \sin(2\theta) i \right)$$

If $z^2 = i$, we must have

$$r^2 = 1$$
$$\cos(2\theta) = \cos \left( \frac{\pi}{2} \right)$$
$$\sin(2\theta) = \sin \left( \frac{\pi}{2} \right)$$

The first of these three equations deals only with real numbers, and there is only one such solution to $r^2 = 1$, namely $r = 1$. There is also an obvious solution to the next two equations: we want

$$2\theta = \frac{\pi}{2}$$

we get a solution

$$\theta = \frac{\pi}{4}.$$ 

However, there is a less obvious second solution. Since $\cos(A + 2\pi) = \cos(A)$ and $\sin(A + 2\pi) = \sin(A)$ for any angle $A$, we can also try to find an angle $\theta$ such that

$$2\theta = \frac{\pi}{2} + 2\pi.$$ 

Thus

$$\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

is also a solution. Thus the two square roots of $i$ are

$$1 \left( \cos \left( \frac{\pi}{4} \right) + \sin \left( \frac{\pi}{4} \right) i \right) = \left( \frac{\sqrt{2}}{2} \right) + \left( \frac{\sqrt{2}}{2} \right) i$$

and

$$1 \left( \cos \left( \frac{5\pi}{4} \right) + \sin \left( \frac{5\pi}{4} \right) i \right) = -\left( \frac{\sqrt{2}}{2} \right) - \left( \frac{\sqrt{2}}{2} \right) i$$
Now let us solve the general problem: $z$ is a non-zero complex number, and we want to find $N^{th}$-roots of $z$. We write $z$ in terms of polar coordinates:

$$z = r \left( \cos(\theta) + \sin(\theta)i \right).$$

We can also write the unknown $w$ in terms of polar coordinates:

$$w = R \left( \cos(\varphi) + \sin(\varphi)i \right).$$

Then

$$w^N = R^N \left( \cos(N\varphi) + \sin(N\varphi)i \right).$$

In order for this to equal $z$ we need

$$R^N = r$$

$$\cos(N\varphi) = \cos(\theta)$$

$$\sin(N\varphi) = \sin(\theta).$$

Since $r > 0$, we can always find a positive real number $R$ so that $R^N = r$. To solve the two trigonometric equations, remember that sin and cos are periodic with period $2\pi$. Thus $\varphi$ will be a solution if

$$N\varphi = \theta + 2\pi k$$

or

$$\varphi = \frac{\theta}{N} + 2\pi \frac{k}{N}$$

where $k$ is any integer. It follows that we get $N$ distinct solutions by taking

$$\varphi = \frac{\theta}{N} \text{ or } \varphi = \frac{\theta}{N} + 2\pi \frac{1}{N} \text{ or } \varphi = \frac{\theta}{N} + 2\pi \frac{2}{N} \text{ or } \ldots \text{ or } \varphi = \frac{\theta}{N} + 2\pi \frac{N-1}{N}.$$

### 1.5. Solving polynomial equations.

#### 1.5.1. Quadratic Equations.

The quadratic equation $az^2 + bz + c = 0$ always has a solution in the complex numbers, and in fact has two solutions if $a \neq 0$. This follows from the quadratic formula, which is derived as follows:

$$az^2 + bz + c = a \left( z^2 + \frac{b}{a}z \right) + c$$

$$= a \left( z^2 + \frac{b}{a}z + \frac{b^2}{4a^2} \right) + c - \frac{b^2}{4a}$$

$$= a \left( z + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}$$
Thus
\[ az^2 + bz + c = 0 \iff a \left( z + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a} = 0 \]
\[ \iff a \left( z + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a} \]
\[ \iff \left( z + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2} \]
\[ \iff \left( z + \frac{b}{2a} \right) = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \]
\[ \iff z = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}. \]

1.5.2. Cubic Equations.

There is also a formula for solving cubic equations
\[ az^3 + bz^2 + cz + d = 0. \] (6)

However, the derivation is much more complicated.

**Step 1:** If \( a \neq 0 \), we can always divide through by \( a \) so that the coefficient of \( z^3 \) is 1. Thus we can assume that we want to solve
\[ z^3 + bz^2 + cz + d = 0. \]

**Step 2:** We try to find a number \( \lambda \in \mathbb{C} \) so that if we replace \( z \) by \( w - \lambda \), the new cubic equation in \( w \) does not have any quadratic term. We get
\[
(w - \lambda)^3 + b(w - \lambda)^2 + c(w - \lambda) + d
= (w^3 - 3\lambda w^2 + 3\lambda^2 w - \lambda^3) + b(w^2 - 2\lambda w + \lambda^2) + c(w - \lambda) + d
= w^3 + (b - 3\lambda)w^2 + (c - 2b\lambda + 3\lambda^2)w + (d - c\lambda + b\lambda^2 - \lambda^3)
\]
Thus if we choose \( \lambda = \frac{b}{3} \), the new equation becomes
\[
w^3 + \left( c - \frac{2}{3} b^2 \right) w + \left( d - \frac{1}{3} bc + \frac{1}{9} b^3 - \frac{1}{27} b^3 \right) = 0. \] (7)

If \( w \) is a root of (7), then \( z = w + \frac{b}{3} \) is a root of (6).

**Step 3:** We write our cubic equation as
\[ w^3 + C w + D = 0, \] (8)
so that
\[
C = c - \frac{2}{3} b^2,
D = d - \frac{1}{3} bc + \frac{1}{9} b^3 - \frac{1}{27} b^3. \]
Now comes the real trick! We let
\[ w = u - \frac{C}{3u}. \]

Then we get
\[
0 = \left( u - \frac{C}{3u} \right)^3 + C \left( u - \frac{C}{3u} \right) + D
\]
\[
= u^3 - 3u^2 \left( \frac{C}{3u} \right) + 3u \left( \frac{C}{3u} \right)^2 - \left( \frac{C}{3u} \right)^3 + Cu - \frac{C^2}{3u} + D
\]
\[
= u^3 - Cu + \frac{C^2}{3u} - \frac{C^3}{27u^3} + Cu - \frac{C^2}{3u} + D
\]
\[
= u^3 - \frac{C^3}{27u^3} + D
\]
\[
= \frac{1}{u^3} \left( (u^3)^2 + D(u^3) - \frac{C^3}{27} \right)
\]
or
\[
0 = \left( u^3 \right)^2 + D \left( u^3 \right) - \frac{C^3}{27}
\]
This is a quadratic equation in \( u^3 \), so we can solve and get
\[
u^3 = \frac{-D \pm \sqrt{D^2 + \frac{4}{27}C^3}}{2} = -\frac{1}{2}D \pm \sqrt{\frac{D^2}{4} + \frac{C^3}{27}}
\]
Thus
\[
u = 3 \sqrt{-\frac{1}{2}D \pm \sqrt{\frac{D^2}{4} + \frac{C^3}{27}}}
\]
and
\[
w = 3 \sqrt{-\frac{1}{2}D \pm \sqrt{\frac{D^2}{4} + \frac{C^3}{27}}} - \frac{C}{3 \sqrt{-\frac{1}{2}D \pm \sqrt{\frac{D^2}{4} + \frac{C^3}{27}}}}
\]
so
\[
z = b \frac{3}{3} + 3 \sqrt{-\frac{1}{2}D \pm \sqrt{\frac{D^2}{4} + \frac{C^3}{27}}} - \frac{C}{3 \sqrt{-\frac{1}{2}D \pm \sqrt{\frac{D^2}{4} + \frac{C^3}{27}}}}
\]