Groups Acting on Surfaces: Finding Global Fixed Points

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Group Actions

Definition. *An action of a group G on a space X is a continuous (or differentiable) function $\phi : G \times X \to X$ satisfying*

1. $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$

2. $\phi(e, x) = x$ *for all* x *where* e *is the identity of* G*.*

A homeomorphism $f : X \to X$ defines an action of Z on on X by $\phi(n, x) = f^n(x)$.

We will be interested in actions of discrete non-compact groups such as $SL(n, \mathrm{Z})$ is the group of $n \times n$ integer matrices with determinant 1.

A Motivating Conjecture

Conjecture. *[R. Zimmer [17]] Any C^∞ volume preserving action of $SL(n, \mathbb{Z})$ on a compact manifold with dimension less than n, factors through an action of a finite group.*

We are really interested in results valid for all finite index subgroups of $SL(n, \mathbb{Z})$.

Theorem. *[D. Witte [16]] Let $\mathcal{G}$ be a finite index subgroup of $SL(n, \mathbb{Z})$ with $n \geq 3$. Any homomorphism*

$$\phi : \mathcal{G} \to \mathrm{Homeo}(S^1)$$

has a finite image.

Example. *The group $SL(3, \mathrm{Z})$ acts analytically on S^2 by projectivizing the standard action on R^3.*

S^2 is the set of unit vectors in R^3. If $x \in S^2$ and $g \in SL(3, \mathrm{Z})$, we can define $\phi(g) : S^2 \to S^2$ by

$$\phi(g)(x) = \frac{gx}{|gx|}.$$

Question. *Let $\mathcal{G}$ be a finite index subgroup of $SL(4, \mathrm{Z})$. Does every homomorphism from $\mathcal{G}$ to $\mathrm{Diff}(S^2)$ or $\mathrm{Homeo}(S^2)$ have a finite image? What about other surfaces?*

The Heisenberg group

Example. *The group of integer matrices of the form*

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

is called the Heisenberg group*.*

If

$$g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

then their *commutator* $f = [g, h] := g^{-1}h^{-1}gh$ is

$$f = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ and f commutes with g and h.

This implies

$$[g^n, h^n] = f^{n^2}.$$

Distortion in Groups

Definition. *[Gromov] An element g in a finitely generated group G is called a* distortion element *if it has infinite order and*

$$\liminf_{n\to\infty} \frac{|g^n|}{n} = 0,$$

where $|g|$ denotes the minmal word length of g in some set of generators. If $\mathcal{G}$ is not finitely generated then g is distorted if it is distorted in some finitely generated subgroup.

Example. *In the subgroup G of $SL(2, \mathrm{R})$ generated by*

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1}BA = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = B^4 \text{ and } A^{-n}BA^n = B^{4^n}$$

so B is distorted.

Example. *In the Heisenberg group the identity*

$$[g^n, h^n] = f^{n^2}.$$

shows f *is distorted since it implies* $|f^{n^2}| \leq 4n$.

Example. *[G. Mess [11]] Consider the subgroup of* $\mathrm{Diff}_\omega(\mathrm{T}^2)$ *generated by the automorphism given by*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and a translation $T(x) = x + w$ *where* $w \neq 0$ *is parallel to the unstable manifold of* A*. The element* T *is distorted.*

Proof:
Let λ be the expanding eigenvalue of A. The element $h_n = A^n T A^{-n}$ satisfies $h_n(x) = x + \lambda^n w$ and $g_n = A^{-n} T A^n$ satisfies $g_n(x) = x + \lambda^{-n} w$. Hence $g_n h_n(x) = x + (\lambda^n + \lambda^{-n})w$. Since $tr A^n = \lambda^n + \lambda^{-n}$ is an integer we conclude $T^{tr A^n} = g_n h_n$, so $|T^{tr A^n}| \leq 4n + 2$. Thus

$$\lim_{n \to \infty} \frac{|T^{tr A^n}|}{tr A^n} = 0,$$

so T is distorted.

Question. *Can one characterize the dynamics of distortion elements in* $\mathrm{Homeo}(S^1)$ *or* $\mathrm{Diff}(S^2)$ *or in area preserving diffeomorphisms of* S^2? *What about irrational rotations of* S^1 *or* S^2 *in the area preserving or analytic case.*

Theorem. [D. Calegari] *There is a* C^0 *action of the Heisenberg group on* S^2 *whose center generated by an irrational rotation. An irrational rotation of* S^2 *is distorted in* $\mathrm{Diff}^\infty(S^2)$*.*

The example of Calegari for the Heisenberg group acting on S^2 is not conjugate to a C^1 example.

Theorem. [D. Calegari] *An irrational rotation of* S^1 *is distorted in* $\mathrm{Diff}^1(S^1)$*.*

Many Lattices have Distortion

Theorem. *[Lubotzky-Mozes-Ragunathan [9]] Suppose* Γ *is a non-uniform irreducible lattice in a semi-simple Lie group* $\mathcal{G}$ *with* $\mathrm{R}-$*rank* ≥ 2. *Suppose further that* $\mathcal{G}$ *is connected, with finite center and no nontrivial compact factors. Then* Γ *has distortion elements, in fact, elements whose word length growth is at most logarithmic.*

Margulis' normal subgroup theorem

Definition. *A group is called* almost simple *if every normal subgroup is finite or has finite index.*

Theorem. *[Margulis] Assume Γ is an irreducible lattice in a semi-simple Lie group with $\mathrm{R}-$rank ≥ 2, e.g. any finite index subgroup of $SL(n, \mathrm{Z})$ with $n \geq 3$. Then any normal subgroup of Γ is either finite and in the center of Γ or has finite index. In particular Γ is almost simple.*

Proposition. *If $\mathcal{G}$ is a finitely generated almost simple group which contains a distortion element and $\mathcal{H} \subset \mathcal{G}$ is a normal subgroup, then the only homomorphism from $\mathcal{H}$ to R is the trivial one.*

Thurston's stability theorem

Theorem. *[Thurston [15]] Suppose $\mathcal{G}$ is a finitely generated group,*

$$\phi : G \to \mathrm{Diff}^1(M^n)$$

is a homomorphism and there is $x_0 \in M$ such that for all $g \in \mathcal{G}$

$$\phi(g)(x_0) = x_0 \text{ and } D\phi(g)(x_0) = I.$$

Then either ϕ is trivial or there is a non-trivial homomorphism from $\mathcal{G}$ to R.

Proof of Thurston's stability theorem

The proof we give is due to W. Schachermayer [14]. Let $\{g_i\}$ be a set of generators for $\phi(\mathcal{G})$. WLOG assume $M = \mathrm{R}^m$ and $x_0 = 0$ is not in the interior of $\mathrm{Fix}(\phi(\mathcal{G}))$.

For $g \in \phi(\mathcal{G})$ let $\widehat{g}(x) = g(x) - x$, so $g(x) = x + \widehat{g}(x)$ and $D\widehat{g}(0) = 0$. We compute

$$\begin{aligned}
\widehat{gh}(x) &= g(h(x)) - x \\
&= h(x) - x + g(h(x)) - h(x) \\
&= \widehat{h}(x) + \widehat{g}(h(x)) \\
&= \widehat{h}(x) + \widehat{g}(x + \widehat{h}(x)) \\
&= \widehat{g}(x) + \widehat{h}(x) + \big(\widehat{g}(x + \widehat{h}(x)) - \widehat{g}(x)\big).
\end{aligned}$$

Hence for all $g, h \in \mathcal{G}$ and for all $x \in \mathrm{R}^m$

$$\widehat{gh}(x) = \widehat{g}(x) + \widehat{h}(x) + \big(\widehat{g}(x + \widehat{h}(x)) - \widehat{g}(x)\big). \quad (1)$$

Choose a sequence $\{x_n\}$ in R^m converging to 0 such that for some i we have $|\widehat{g}_i(x_n)| \neq 0$ for all n. Possible since 0 is not in the interior of $\mathrm{Fix}(\phi(\mathcal{G}))$.

Let $M_n = \max\{|\widehat{g}_1(x_n)|, \ldots, |\widehat{g}_k(x_n)|\}$. Passing to a subsequence we may assume that for each i the limit

$$L_i = \lim_{n\to\infty} \frac{\widehat{g}_i(x_n)}{M_n}$$

exists and that $\|L_i\| \leq 1$. For some i we have $\|L_i\| = 1$; say for $i = 1$.

If $g \in \mathcal{G}$ and the limit

$$L = \lim_{n\to\infty} \frac{\widehat{g}(x_n)}{M_n}$$

exists then for each i we will show that

$$\lim_{n\to\infty} \frac{\widehat{g_i g}(x_n)}{M_n} = L_i + L. \qquad (2)$$

By Equation (1) it suffices to show

$$\lim_{n\to\infty} \frac{\widehat{g}_i(x_n + \widehat{g}(x_n)) - \widehat{g}_i(x_n)))}{M_n} = 0. \qquad (3)$$

By the mean value theorem

$$\lim_{n\to\infty} \left\| \frac{\widehat{g}_i(x_n + \widehat{g}(x_n)) - \widehat{g}_i(x_n)))}{M_n} \right\|$$
$$\leq \lim_{n\to\infty} \sup_{t\in[0,1]} \|D\widehat{g}_i(z_n(t))\| \left\| \frac{\widehat{g}(x_n)}{M_n} \right\|,$$

where $z_n(t) = x_n + t\widehat{g}(x_n)$. But

$$\lim_{n\to\infty} \frac{\widehat{g}(x_n)}{M_n} = L \text{ and } \lim_{n\to\infty} sup_{t\in[0,1]} \|D\widehat{g}_i(z_n(t))\| = 0,$$

so Equation (3) holds. Defining $\Theta : \phi(\mathcal{G}) \to \mathrm{R}^m$ by

$$\Theta(g) = \lim_{n\to\infty} \frac{\widehat{g}(x_n)}{M_n}$$

gives a homomorphism from $\phi(\mathcal{G})$ to R^m.

N.B.: For definitive results on C^1 actions on S^1 see E. Ghys, [7].

Theorem. [Toy Theorem] *Suppose $\mathcal{G}$ is a finitely generated almost simple group and has a distortion element and suppose μ is a finite probability measure on S^1. If*

$$\phi : \mathcal{G} \to Diff_\mu(S^1)$$

is a homomorphism then $\phi(\mathcal{G})$ is finite.

Proof:

- The rotation number $\rho : Diff_\mu(S^1) \to \mathrm{R}/\mathrm{Z}$ is a homomorphism.

- If f is distorted $\rho(f^n) = 0$ for some $n > 0$ so $\mathrm{Fix}(f^n)$ is non-empty.

- $supp(\mu) \subset \mathrm{Fix}(f^n)$

- $\mathcal{G}_0 := \{g \in \mathcal{G} \mid \phi(g)$ pointwise fixes $supp(\mu)\}$ is infinite and normal, and hence finite index.

- $\phi(\mathcal{G}_0)$ is trivial by Thurston stability.

Theorem. *[F-Handel [4]] Suppose S is a closed oriented surface of genus at least one and μ is a Borel probabilty measure on S with infinite support. Suppose $\mathcal{G}$ is finitely generated, almost simple and has a distortion element. Then any homomorphism*

$$\phi : \mathcal{G} \to \mathrm{Diff}_{\mu}(S)$$

has finite image.

This result was previously known in the special case of symplectic diffeomorphisms and Lebesgue measure by a result of L. Polterovich [13].

The result above also holds even when $supp(\mu)$ is finite if $\mathcal{G}$ is a Kazhdan group (aka $\mathcal{G}$ has property T).

Distortion and Measure

Theorem. *[F-Handel [4]] Suppose that S is a closed oriented surface, that f is a distortion element in $\mathrm{Diff}(S)_0$ and that μ is an f-invariant Borel probability measure.*

1. *If S has genus at least two then $\mathrm{Per}(f) = \mathrm{Fix}(f)$ and $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f)$.*

2. *If $S = T^2$ and $\mathrm{Per}(f) \neq \emptyset$, then all points of $\mathrm{Per}(f)$ have the same period, say n, and $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f^n)$*

3. *If $S = S^2$ and if f^n has at least three fixed points for some smallest $n > 0$, then $\mathrm{Per}(f) = \mathrm{Fix}(f^n)$ and $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f^n)$.*

Heisenberg again

Theorem. *[F-Handel [4]] Suppose S is a closed oriented surface with Borel probabilty measure μ and $\mathcal{G}$ is a finitely generated, almost simple group with a subgroup isomorphic to the Heisenberg group. Then any homomorphism*

$$\phi : \mathcal{G} \to \mathrm{Diff}_\mu(S)$$

has finite image.

Parallels between $\mathrm{Diff}(S^1)_0$ and $\mathrm{Diff}_\mu(S)_0$

In general there seem to be strong parallels between results about $\mathrm{Diff}(S^1)_0$ and $\mathrm{Diff}_\mu(S)_0$. For example, Witte's theorem and our results above. Also we have

Theorem. *[Hölder] Suppose $\mathcal{G}$ is a subgroup of $\mathrm{Homeo}(S^1)_0$ which acts freely (no non-trivial element has a fixed point). Then $\mathcal{G}$ is Abelian.*

Theorem. *[Conley-Zehnder, Matsumoto] Suppose*

$$f \in \mathrm{Homeo}_\omega(\mathrm{T}^2)_0$$

is a commutator (ω is Lebesgue measure). Then f has (at least three) fixed points.

Corollary. *Suppose $\mathcal{G}$ is a subgroup of $\mathrm{Homeo}_\omega(\mathrm{T}^2)_0$ which acts freely. Then $\mathcal{G}$ is Abelian.*

Nilpotent Groups

Definition. *A group $\mathcal{N}$ is called* nilpotent *provided when we define*

$$\mathcal{N}_0 = \mathcal{N},\ \mathcal{N}_i = [\mathcal{N}, \mathcal{N}_{i-1}],$$

there is an $n \geq 1$ such that $\mathcal{N}_n = \{e\}$. Note if $n = 1$ it is Abelian.

Theorem. *[Plante - Thurston [12]] Let N be a nilpotent subgroup of $\mathrm{Diff}^2(S^1)_0$. Then N must be Abelian.*

Theorem. *[Farb - F] Every finitely-generated, torsion-free nilpotent group is isomorphic to a subgroup of $\mathrm{Diff}^1(S^1)_0$.*

An Analogue of the Plante - Thurston Theorem

Theorem. *[F - Handel] Let $\mathcal{N}$ be a nilpotent subgroup of $\mathrm{Diff}^1_\mu(S)_0$ with μ a probability measure with $supp(\mu) = S$. If $S \neq S^2$ then $\mathcal{N}$ is Abelian, if $S = S^2$ then $\mathcal{N}$ is Abelian or has an index 2 Abelian subgroup.*

Proof: (For the case $genus(S) > 1$) Suppose

$$\mathcal{N} = \mathcal{N}_1 \supset \cdots \supset \mathcal{N}_m \supset \{1\}$$

is the lower central series of $\mathcal{N}$. then $\mathcal{N}_m$ is in the center of $\mathcal{N}$. If $m > 1$ there is a non-trivial $f \in \mathcal{N}_m$ and elements g, h with $f = [g, h]$. No non-trivial element of $\mathrm{Diff}^1(S)_0$ has finite order since S has genus > 1. So g, h generate a Heisenberg group and f is distorted. Our theorem says $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f)$, but $\mathrm{supp}(\mu) = S$ so $f = id$. This is a contradiction unless $m = 1$ and $\mathcal{N}$ is abelian.

Fixed Points for Abelian Actions: S^2

Theorem. *[F, Handel, Parwani [5]] Let $\mathcal{G}$ be an abelian subgroup of $\mathrm{Diff}^1(\mathrm{R}^2)_0$ with the property that there is a compact $\mathcal{G}$ invariant subset of R^2. Then there is a point $x \in \mathrm{R}^2$ such that $g(x) = x$ for all g in $\mathcal{G}$.*

Theorem. *[F, Handel, Parwani [5]] Let $\mathcal{G}$ be an abelian subgroup of $\mathrm{Diff}^1(S^2)_0$ Then there is a subgroup $\mathcal{G}_0$of $\mathcal{G}$ of index at most two and a point $x \in S^2$ such that $g(x) = x$ for all g in $\mathcal{G}_0$.*

The theorem above was previously proved by M. Handel [8] for groups generated by two elements.

Abelian Actions: Genus ≥ 2

Theorem. *[F, Handel, Parwani [6]] Suppose S is a closed oriented surface of genus at least two and that $\mathcal{F}$ is an abelian subgroup of $\mathrm{Diff}_0(S)$ Then the set of contractible fixed points, $\mathrm{Fix_c}(\mathcal{F})$, is non-empty. In particular $\mathrm{Fix}(\mathcal{F})$ is non-empty.*

Theorem. *[F, Handel, Parwani [6]] Suppose S is a closed oriented surface of genus at least two and that $\mathcal{F}$ is an abelian subgroup of $\mathrm{Diff}(S)$. Then $\mathcal{F}$ has a finite index subgroup $\mathcal{F}_0$ such that $\mathrm{Fix}(\mathcal{F}_0)$ is non-empty.*

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