

Hecke algebras and spherical functions

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Let

G be a group and K a subgroup.

The Hecke algebra is

$$H = \text{End}_G(\text{Ind}_K^G(\mathbf{1})).$$

Utility: As a (G, H) bimodule

$$\text{Ind}_K^G(\mathbf{1}) \cong G^\lambda \otimes H^\lambda \quad \text{where} \quad \begin{array}{l} G^\lambda \text{ is an irreducible } G\text{-module,} \\ H^\lambda \text{ is an irreducible } H\text{-module.} \end{array}$$

Alternatively, there are well behaved functors

$$\begin{array}{ccc} \{G\text{-modules}\} & \longleftrightarrow & \{H\text{-modules}\} \\ G^\lambda & \longleftrightarrow & H^\lambda \end{array} = \text{Hom}_G(\text{Ind}_K^G(\mathbf{1}), G^\lambda)$$

so that one can transfer back and forth between the representation theory of G and the representation theory of H .

On characters: If $\chi \in Z(\mathbb{C}G)$, the center of the group algebra of G , then the action of χ on $\text{Ind}_K^G(\mathbf{1})$ produces an element $\varphi(\chi) \in Z(H)$,

$$\begin{array}{ccc} Z(\mathbb{C}G) & \longrightarrow & Z(H) \\ \chi^\lambda & \longmapsto & z_\lambda \end{array}$$

which sends an irreducible character χ^λ of G to the corresponding *spherical function* z_λ in $Z(H)$.

1 What does H look like?

$\mathbf{1} = \mathbb{C} \cdot v_1$ is a K -module via

$$kv_1 = v_1, \quad \text{for } k \in K.$$

Then

$$\begin{aligned}\text{Ind}_K^G(\mathbf{1}) &= \mathbb{C}G \otimes_{\mathbb{C}K} \mathbf{1} = \mathbb{C}[G/K] \\ &= \{f: G \rightarrow \mathbb{C} \mid f(gk) = f(g) \text{ for all } k \in K\},\end{aligned}$$

with G action given by

$$(gf)(h) = f(g^{-1}h), \quad \text{for all } g, h \in G.$$

G acts on G by multiplication. Since *left multiplication* commutes with *right multiplication*,

$$H = \mathbb{C}[K \backslash G / K] = \{f: G \rightarrow \mathbb{C} \mid f(k_1 g k_2) = f(g) \text{ for all } k_1, k_2 \in K\}$$

with product *convolution*,

$$(f_1 f_2)(g) = \int_G f_1(gh^{-1}) f_2(h) dh.$$

If

$$G = \bigsqcup_{w \in W} KwK \quad \text{then} \quad H \text{ has basis } \{T_w \mid w \in W\},$$

where T_w is the characteristic function of the double coset KwK .

2 Example 1

Let

$$\mathcal{V} = \{\text{subspaces of } \mathbb{F}_q^n\} \quad \text{ordered by inclusion.}$$

A maximal chain in this poset is a *flag*

$$0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n, \quad \text{where} \quad \dim(V_i) = i.$$

If

$$G = GL_n(\mathbb{F}_q) \quad \text{and} \quad B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$$

is the subgroup of upper triangular matrices, then

$$\begin{aligned} G/B &\xrightarrow{\sim} \{\text{flags}\} \\ gB &\mapsto g(0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots) \end{aligned} \quad \text{where} \quad e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence

$$\text{Ind}_B^G(\mathbf{1}) = \text{span}\{\text{flags}\}.$$

A basic theorem in linear algebra says that the permutations are representatives of the double cosets in $B \backslash / B$,

$$G = \bigsqcup_{w \in S_n} BwB \quad \text{and so} \quad H = \text{span}\{T_w \mid w \in W\}.$$

where B is as in Example 1. Then

K is a maximal compact subgroup of G and I is an *Iwahori subgroup*.

Then

$$G = \bigsqcup_{\lambda \in P^+} Kt_\lambda K, \quad \text{and} \quad G = \bigsqcup_{\lambda \in \tilde{S}_n} IwI,$$

where

$$t_\lambda = \begin{pmatrix} p^{\lambda_1} & & & & \\ & p^{\lambda_2} & & & \\ & & \ddots & & \\ & & & p^{\lambda_n} & \\ & & & & \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 0 & 0 & 0 & p^{\lambda_3} & 0 \\ 0 & p^{\lambda_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p^{\lambda_1} \\ p^{\lambda_4} & 0 & 0 & 0 & 0 \\ 0 & 0 & p^{\lambda_2} & 0 & 0 \end{pmatrix}.$$

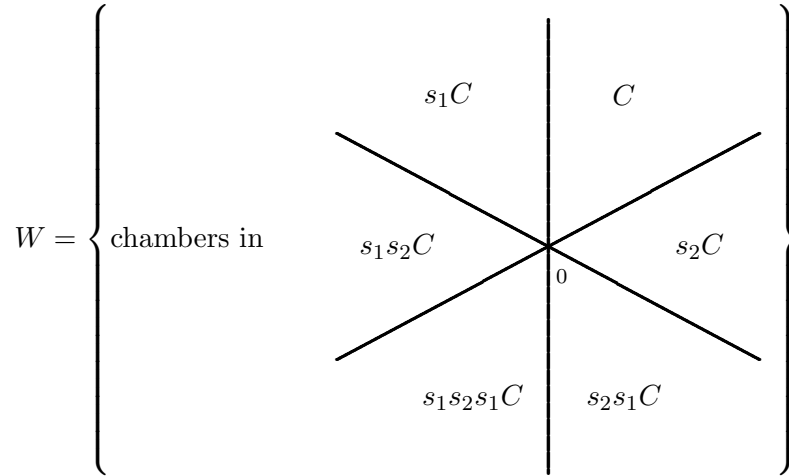
5 Example 3b

$$\begin{array}{ccc} G = & SL_3(\mathbb{Q}_p) & \\ & \cup & \\ K = & SL_3(\mathbb{Z}_p) & \xrightarrow{\Phi} SL_3(\mathbb{F}_p) = G^F \\ & \cup & \\ I = & \Phi^{-1}(B) & \longrightarrow B \end{array}$$

Then

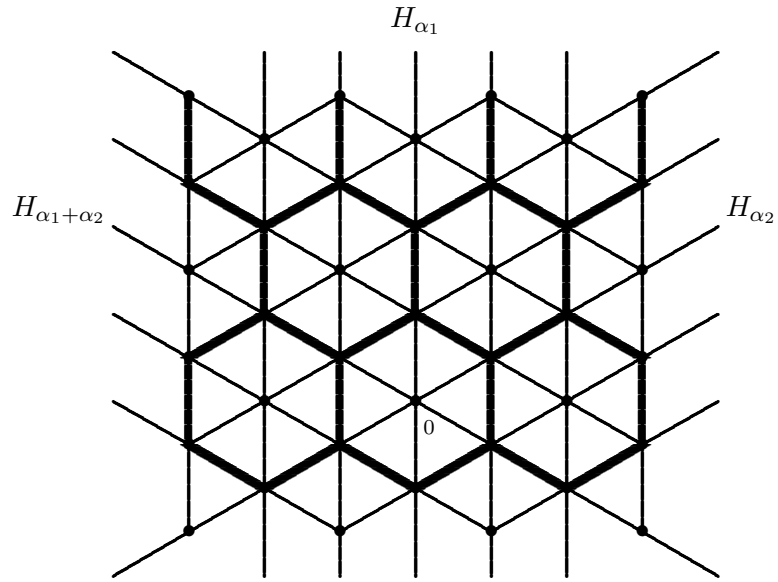
$$G^F = \bigsqcup_{w \in W} BwB, \quad G = \bigsqcup_{\lambda \in P^+} Kt_\lambda K, \quad G = \bigsqcup_{\lambda \in \tilde{W}} IwI,$$

where



and

$$\begin{aligned} \tilde{W} &= \{\text{alcoves in the following picture}\} \\ P &= \{(\text{centers of}) \text{ hexagons in following picture}\} \end{aligned}$$



and $P^+ = P \cap \overline{C}$ where \overline{C} is the closure of the chamber C .

Problem: Give a formula for the spherical functions of G/K in terms of this notation.

This problem was solved by C. Schwer, and my recent preprint *Alcove walks, Hecke algebras, crystals and column strict tableaux* is a further development of the technique of Schwer.