The Schur Hopf algebra

Combinatorics seminar
University of Wisconsin, Madison
February 27, 2006

Arun Ram
Department of Mathematics
University of Wisconsin
Madison, WI 53706
ram@math.wisc.edu

1 Shuffles and splittings

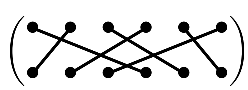
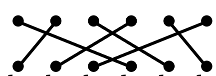
The *letters* are the elements of the *alphabet*

$$B = \{b_1 \ldots, b_n\} \qquad \text{and} \qquad B^{\otimes k} = \{b_{i_1} \cdots b_{i_k} \mid 1 \le i_1, \ldots, i_k \le n\}$$

is the set of *words of length k*.
The symmetric group S_k acts on $B^{\otimes k}$ by

$$\sigma(b_{i_1} \cdots b_{i_k}) = b_{i_{\sigma(1)}} \cdots b_{i_{\sigma(k)}}.$$

Example:

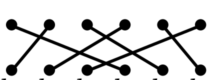
 $b_1b_1b_2b_1b_2b_3 =$ 

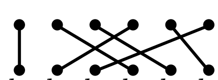
$= b_1b_1b_2b_1b_3b_2.$

The *shuffles* of $b_{i_1} \cdots b_{i_k}$ and $b_{j_1} \cdots b_{j_\ell}$ are the elements of the multiset

$$b_{i_1} \cdots b_{i_k} \sqcup\!\sqcup\, b_{j_1} \cdots b_{j_\ell} = \{\gamma(b_{i_1} \cdots b_{i_k} b_{j_1} \cdots b_{j_\ell})) \mid \gamma \in S_{k\ell}/S_k \times S_\ell\}.$$

Example: Two shuffles in $b_1b_1b_2 \sqcup\!\sqcup\, b_1b_2b_3$ are



$= b_1b_1b_2b_1b_3b_2$ and 

$= b_1b_1b_2b_1b_3b_2.$

The *splittings* of $b_{i_1} \cdots b_{i_k}$ are

$$b_{i_1} \cdots b_{i_r} \otimes b_{i_{r+1}} \cdots b_{i_k}, \qquad \text{for} \quad 0 \le r \le k.$$

2 The Malvenuto-Reutenauer Hopf algebra

$$\mathbb{C}S = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \mathbb{C}S_k$$

has two products and two coproducts

$$\sigma \bullet \tau = \sum_{\gamma} \gamma, \qquad \text{the sum of the shuffles of the top vertices of } \sigma\otimes\tau,$$

$$\sigma * \tau = \sum_{\delta} \delta, \qquad \text{the sum of the shuffles of the bottom vertices of } \sigma\otimes\tau,$$

$$\Delta^\bullet(\sigma) = \sum_{\sigma} \sigma_{(1)}\otimes\sigma_{(2)}, \qquad \text{the sum of the splittings of the bottom vertices of } \sigma,$$

$$\Delta^*(\sigma) = \sum_{\sigma} \sigma^{(1)}\otimes\sigma^{(2)}, \qquad \text{the sum of the splittings of the top vertices of } \sigma.$$

Theorem 2.1. *$(\mathbb{C}S, m^\bullet, \Delta^\bullet)$ and $(\mathbb{C}S, m^*, \Delta^*)$ are dual Hopf algebras and*

$$\begin{array}{ccc} (\mathbb{C}S, m^\bullet, \Delta^\bullet) & \xrightarrow{\sim} & (\mathbb{C}S, m^*, \Delta^*) \\ \sigma & \longrightarrow & \sigma^{-1} \end{array}$$

is a Hopf algebra isomorphism.

3 The tensor Hopf algebra

$$V^{\otimes} = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} V^{\otimes k}$$

with product

$$(b_{i_1}\cdots b_{i_k})(b_{j_1}\cdots b_{j_\ell}) = b_{i_1}\cdots b_{i_k}b_{j_1}\cdots b_{j_\ell}$$

and coproduct the algebra homomrphism $\Delta\colon V^{\otimes}\to V^{\otimes}\otimes V^{\otimes}$ given by

$$\Delta(b_i) = b_i\otimes 1 + 1\otimes b_i, \qquad \text{for} \quad 1\leq i\leq n.$$

Then $V^{\otimes}$ is the universal enveloping algebra of the free Lie algebra on the set B.

$$\begin{aligned}\Delta(b_{i_1}\cdots b_{i_k}) &= (b_{i_1}\otimes 1 + 1\otimes b_{i_1})\cdots(b_{i_k}\otimes 1 + 1\otimes b_{i_k}) \\ &= \sum_{b_{i_1}\cdots b_{i_k}\in u\sqcup\sqcup v} u\otimes v = \sum_{\ell=0}^{k}\sum_{\gamma\in S_k/S_\ell\times S_{k-\ell}} \gamma^{-1}(b_{i_1}\cdots b_{i_k}).\end{aligned}$$

4 The shuffle Hopf algebra

The *shuffle Hopf algebra* is the dual of the tensor Hopf algebra, i.e.

$$V^{\otimes} = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} V^{\otimes k}$$

with product

$$(b_{i_1}\cdots b_{i_k}) \sqcup\sqcup (b_{j_1}\cdots b_{j_\ell}) = \sum_{\gamma\in S_{k+\ell}/S_k\times S_\ell} \gamma(b_{i_1}\cdots b_{i_k}b_{j_1}\cdots b_{j_\ell})$$

and coproduct

$$\Delta^{\sqcup\sqcup}(b_{i_1}\cdots b_{i_k}) = \sum_{r=0}^{k} b_{i_1}\cdots b_{i_r}\otimes b_{i_{r+1}}\cdots b_{i_k}.$$

5 The convolution algebra

The *convolution algebra* is

$$E = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \mathrm{End}(V^{\otimes k})$$

with products

composition $\varphi \circ \psi$,

tensor convolution $\varphi \bullet \psi = m^\bullet \circ (\varphi \otimes \psi) \circ \Delta^\bullet$,

shuffle convolution $\varphi * \psi = m^{\sqcup\sqcup} \circ (\varphi \otimes \psi) \circ \Delta^{\sqcup\sqcup}$.

The vector space $\mathrm{End}(V^{\otimes k})$ has basis

$$\left\{\varphi_{[\gamma]} \;\middle|\; [\gamma] = \begin{bmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{bmatrix},\quad 1 \leq i_1, \ldots, i_k, j_1, \ldots, j_k \leq n\right\}$$

where the $(i'_1 \ldots i'_k, j'_1 \ldots j'_k)$ entry of $\varphi_{[\gamma]}$ is

$$(\varphi_{[\gamma]})^{i'_1\ldots i'_k}_{j'_1\ldots j'_k} = \begin{cases} 1, & \text{if } \begin{bmatrix} i'_1 & \cdots & i'_k \\ j'_1 & \cdots & j'_k \end{bmatrix} = \begin{bmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{bmatrix}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{bmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{bmatrix} \bullet \begin{bmatrix} i'_1 & \cdots & i'_\ell \\ j_1 & \cdots & j'_\ell \end{bmatrix} = \sum_{r_1\ldots r_{k+\ell}\in(j_1\ldots j_k)\sqcup\sqcup(j'_1\ldots j'_\ell)} \begin{bmatrix} i_1 & \cdots & i_k & i'_1 & \cdots & i'_\ell \\ r_1 & \cdots & & & \cdots & r_{k+\ell} \end{bmatrix},$$

(shuffle the bottom and concatenate the top), and

$$\begin{bmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{bmatrix} * \begin{bmatrix} i'_1 & \cdots & i'_\ell \\ j_1 & \cdots & j'_\ell \end{bmatrix} = \sum_{s_1\ldots s_{k+\ell}\in(j_1\ldots j_k)\sqcup\sqcup(j'_1\ldots j'_\ell)} \begin{bmatrix} s_1 & \cdots & & & \cdots & s_{k+\ell} \\ j_1 & \cdots & j_k & j'_1 & \cdots & j'_\ell \end{bmatrix},$$

(shuffle the top and concatenate the bottom).

6 Centralizer algebras

The group GL_n acts on $V^{\otimes k}$ by

$$gb_i = \sum_{j=1}^{n} g_{ji} b_j \qquad \text{and} \qquad g(b_{i_1} \cdots b_{i_k}) = (gb_{i_1}) \cdots (gb_{i_k}),$$

for $g = (g_{ij})$ in GL_n. Then

$$\mathrm{End}_{GL_n}(V^{\otimes k}) \cong \mathbb{C}S_k, \qquad \text{for } k \leq n,$$

and the products and coproducts on $\mathbb{C}S$ are

$m^\bullet$ the restriction of tensor convolution,

$m^{\sqcup\sqcup}$, the restriction of shuffle convolution,

$\Delta^{\bullet}$, the dual of $m^{\sqcup\sqcup}$,

$\Delta^{\sqcup\sqcup}$, the dual of $m^{\bullet}$.

The *Schur algebra*

$$U_k = \mathrm{End}_{S_k}(V^{\otimes k})$$

has basis

$$\left\{\varphi_{[\gamma]} \;\middle|\; \{\gamma\} = \begin{Bmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{Bmatrix},\quad 1 \le i_1, \ldots, i_k, j_1, \ldots, j_k \le n\right\}$$

where the $(i'_1 \ldots i'_k, j'_1 \ldots j'_k)$ entry of $\varphi_{\{\gamma\}}$ is

$$\left(\varphi_{\{\gamma\}}\right)^{i'_1 \ldots i'_k}_{j'_1 \ldots j'_k} = \begin{cases} 1, & \text{if } \begin{Bmatrix} i'_1 & \cdots & i'_k \\ j'_1 & \cdots & j'_k \end{Bmatrix} = \begin{Bmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{Bmatrix}, \\ 0, & \text{otherwise.} \end{cases}$$