

\mathfrak{p} -adic groups

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1 \mathfrak{p} -adic groups

Let \mathbb{F} be \mathfrak{p} -adic field so that

$$\begin{array}{ccc} \mathbb{F} & & \\ \cup | & & \text{where } k = \mathfrak{o}/\mathfrak{p}. \\ \mathfrak{o} & \longrightarrow & k \end{array}$$

and our favourite example is

$$\begin{aligned} \mathbb{F} &= \mathbb{C}((t)) = \{ct^k(1 + a_1t + a_2t^2 + \dots) \mid c \in \mathbb{C}^*, k \in \mathbb{Z}, a_i \in \mathbb{C}\}, \\ \mathfrak{o} &= \mathbb{C}[[t]], \\ k &= \mathbb{C}, \end{aligned}$$

with

$$\begin{aligned} \mathfrak{p} &= (t) = t\mathbb{C}[[t]] = \{a_1t + a_2t^2 + \dots \mid a_i \in \mathbb{C}\}, \quad \text{and} \\ \mathfrak{o}^* &= \{c(1 + a_1t + \dots) \mid c \in \mathbb{C}^*, a_i \in \mathbb{C}\}. \end{aligned}$$

Let $G(\mathbb{F})$ be a \mathfrak{p} -adic group so that

$$\begin{array}{ccc} G & = & G(\mathbb{F}) \\ \cup | & & \\ K & = & G(\mathfrak{o}) \longrightarrow G(k) \\ \cup | & & \cup | \\ I & \longrightarrow & B(k) \end{array}$$

where our favourite example is

$$\begin{array}{l} G = GL_n(\mathbb{C}((t))) \\ \cup \\ K = GL_n(\mathbb{C}[[t]]) \longrightarrow GL_n(\mathbb{C}) \\ \cup \\ I \longrightarrow \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \end{array}$$

2 Root structure

$GL_n(\mathbb{F})$ is generated by

$$x_{ij}(a) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & a & \\ & & & 1 \end{pmatrix} = \prod_{k>-\ell} x_{ij}(t^k a_k), \quad \text{if } a = a_{-\ell}t^{-\ell} + a_{-\ell+1}t^{-\ell+1} + \dots$$

and

$$h_i(g) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & g & \\ & & & & 1 \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \quad \text{where } g \in \mathbb{C}((t))^*,$$

and contains a lattice

$$H = \left\{ \begin{pmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_n} \end{pmatrix} \mid \lambda_1, \dots, \lambda_n \in \mathbb{Z} \right\} \quad \text{and} \quad S_n = \{\text{permutation matrices}\}$$

and the affine Weyl group of type GL_n is

$$\tilde{W} = \left\{ \begin{array}{l} \text{matrices with entries } t^k, k \in \mathbb{Z}, \text{ with} \\ \text{exactly one nonzero entry in each row and each column} \end{array} \right\}$$

Let

$$U^+ = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & a_{ij} & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \mid a_{ij} \in \mathbb{C}((t)) \right\} = \left\{ \prod_{1 \leq i < j \leq n} x_{ij}(a_{ij}) \mid a_{ij} \in \mathbb{C}((t)) \right\}.$$

Theorem 2.1.

(a) $GL_n(\mathbb{F}) = \bigsqcup_{w \in \tilde{W}} IwI$

(b) $G = \bigsqcup_{v \in \tilde{W}} U^+vI$

These are analogues of

$$GL_n(\mathbb{C}) = \bigsqcup_{w \in S_n} U^- w B.$$

which is proved by taking a matrix, row reducing to obtain pivots in each row, and then multiplying by a permutation to move the pivots into successive rows so that the resulting matrix is upper triangular. The row reduction is effected by multiplication by generators $x_{ij}(a)$, since

$$x_{ij}A \text{ takes } a \cdot (j^{\text{th}} \text{ row}) \text{ and adds it to the } i^{\text{th}} \text{ row.}$$

3 $SL_2(\mathbb{F})$

Let

$$x_{\alpha_1+k\delta}(c) = \begin{pmatrix} 1 & t^k c \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad x_{-\alpha_1+k\delta}(c) = \begin{pmatrix} 1 & 0 \\ t^k c & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} x_{\alpha_1+k\delta}(c)x_{-\alpha_1-k\delta}(-c^{-1})x_{\alpha_1+k\delta}(c) &= \begin{pmatrix} 0 & t^k c \\ t^k c & 1 \end{pmatrix} \\ x_{-\alpha_1+k\delta}(c)x_{\alpha_1-k\delta}(-c^{-1})x_{-\alpha_1+k\delta}(c) &= \begin{pmatrix} 0 & -t^k c \\ t^k c & 1 \end{pmatrix} \end{aligned}$$

Let

$$\begin{aligned} n_{\alpha_1+k\delta} &= \begin{pmatrix} 0 & t^k \\ -t^{-k} & 0 \end{pmatrix} = \begin{pmatrix} t^k & 0 \\ 0 & t^{-k} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = t_{k\alpha_1} s_1, \\ n_{-\alpha_1+k\delta} &= \begin{pmatrix} 0 & -t^{-k} \\ t^k & 0 \end{pmatrix} = \begin{pmatrix} -t^{-k} & 0 \\ 0 & -t^{-k} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -t_{-k\alpha_1} s_1, \end{aligned}$$

and

$$\begin{aligned} s_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = s_{\alpha_1} \quad \text{with} \quad n_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ s_0 &= \begin{pmatrix} 0 & -t^{-1} \\ t & 0 \end{pmatrix} = s_{\alpha_0} \quad \text{with} \quad n_1 = \begin{pmatrix} 0 & t^{-1} \\ -t & 0 \end{pmatrix} \quad \text{and} \quad \alpha_0 = -\alpha_1 + \delta. \end{aligned}$$

Then

$$s_1 s_0 = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \text{and} \quad s_0 s_1 = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$$

Let

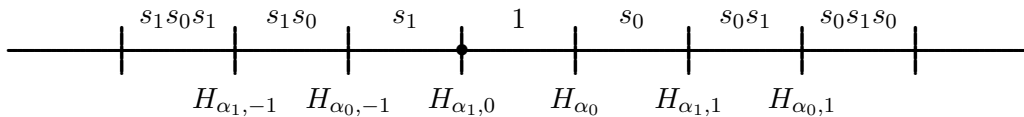
$$n_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad n_0 = \begin{pmatrix} 0 & t^{-1} \\ -t & 0 \end{pmatrix}$$

so that

$$x_{\alpha_1+k\delta}(c)n_1 = n_1 x_{-\alpha_1+k\delta}(-c) \quad \text{and} \quad x_{-\alpha_1+k\delta}(c)n_1 = n_1 x_{\alpha_1+k\delta}(-c),$$

and

$$x_{\alpha_1+k\delta}(c)n_1 = n_0 x_{-\alpha_1+(k+2)\delta}(c) \quad \text{and} \quad x_{-\alpha_1+k\delta}(c)n_1 = n_1 x_{\alpha_1+(k-2)\delta}(c),$$



4 The double cosets IwI

$$x_{\alpha_1}(c_1)s_1I \text{ is in } Is_1I. \quad \begin{array}{c} H_{\alpha_1} \\ s_1 \quad 1 \\ \leftarrow c_1 \end{array}$$

$$x_{-\alpha_1+\delta}(c_1)s_0I \text{ is in } Is_0I. \quad \begin{array}{c} H_{-\alpha_1+\delta} \\ 1 \quad s_0 \\ \rightarrow c_1 \end{array}$$

$$x_{\alpha_1}(c_1)s_1x_{-\alpha_1+\delta}(c_2)s_0I = x_{\alpha_1}(c_1)x_{\alpha_1+\delta}(c_2)s_1s_0I \text{ is in } Is_1s_0I.$$

$$\begin{array}{c} H_{\alpha_1+\delta} \quad H_{\alpha_1} \\ s_1s_0 \quad s_1 \quad 1 \\ \leftarrow c_2 \quad \leftarrow c_1 \end{array}$$

$$x_{-\alpha_1+\delta}(c_1)s_0x_{\alpha_1}(c_2)s_1I = x_{-\alpha_1+\delta}(c_1)x_{-\alpha_1+2\delta}(c_2)s_0s_1I \text{ is in } Is_0s_1I.$$

$$\begin{array}{c} H_{-\alpha_1+\delta} \quad H_{-\alpha_1+2\delta} \\ 1 \quad s_0 \quad s_0s_1 \\ \rightarrow c_1 \quad \rightarrow c_2 \end{array}$$

$$x_{\alpha_1}(c_1)s_1x_{-\alpha_1+\delta}(c_2)s_0x_{\alpha_1}(c_3)s_1I = x_{\alpha_1}(c_1)x_{\alpha_1+\delta}(c_2)x_{\alpha_1+2\delta}(c_3)s_1s_0s_1I \text{ is in } Is_1s_0s_1I.$$

$$x_{-\alpha_1+\delta}(c_1)s_0x_{\alpha_1}(c_2)s_1x_{-\alpha_1+\delta}(c_3)s_0I = x_{-\alpha_1+\delta}(c_1)x_{-\alpha_1+2\delta}(c_2)x_{-\alpha_1+3\delta}(c_3)s_0s_1s_0I \text{ is in } Is_0s_1s_0I.$$

Note that

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} s_1I = \left(\prod_{k \geq 0} x_{\alpha_1+k\delta}(a_k) \right) s_1I = s_1 \left(\prod_{k \geq 0} x_{-\alpha_1+k\delta}(-a_k) \right) I = s_1x_{-\alpha}(-a_0)I = x_{\alpha_1}(a_0)s_1I,$$

and

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} s_1I = \left(\prod_{k \geq 1} x_{-\alpha_1+k\delta}(b_k) \right) s_1I = s_1 \left(\prod_{k \geq 1} x_{\alpha_1+k\delta}(-b_k) \right) I = s_1I,$$

and, in fact

$$Is_1I = \{x_{\alpha_1}(c_1)s_1I \mid c \in CC\},$$

since, for $b \in \mathfrak{p}$,

$$\begin{aligned} x_{-\alpha_1}(b)x_{\alpha_1}(a_0)s_1I &= x_{-\alpha_1}(b)x_{\alpha_1}(a_0)x_{\alpha_1}(-a_0)x_{-\alpha_1}(a_0^{-1})x_{\alpha_1}(-a_0)I \\ &= x_{-\alpha_1}(a_0^{-1})x_{-\alpha_1}(b)I = x_{-\alpha_1}(a_0^{-1})I = x_{\alpha_1}(a_0)s_1I. \end{aligned}$$

A similar computation is

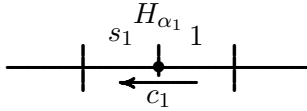
$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} s_0I = \left(\prod_{k \geq 0} x_{\alpha_1+k\delta}(a_k) \right) s_0I = s_1 \left(\prod_{k \geq 0} x_{-\alpha_1+(k+2)\delta}(-a_k) \right) I = s_0I,$$

and

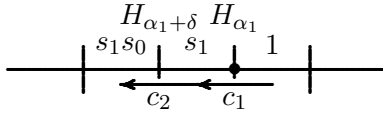
$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} s_0 I &= \left(\prod_{k \geq 1} x_{-\alpha_1 + k\delta}(-b_k) \right) s_0 I = s_0 \left(\prod_{k \geq 1} x_{\alpha_1 + (k-2)\delta}(-b_k) \right) I \\ &= s_0 x_{\alpha_1 - \delta}(-b_1) I = x_{-\alpha_1 + \delta}(b_1) s_0 I, \end{aligned}$$

5 The double cosets $U^+ v I$

$$I \cap U^+ I = I \quad \text{since} \quad I \subseteq U^+ I,$$

$$I s_1 I \cap U^+ s_1 I = \{x_{\alpha_1}(c_1) s_1 I \mid c_1 \in \mathbb{C}\}$$


since $I s_1 I = \{x_{\alpha_1}(c_1) s_1 I \mid c_1 \in \mathbb{C}\} \subseteq U^+ s_1 I$, and

$$I s_1 s_0 I \cap U^+ s_1 I = \{x_{\alpha_1}(c_1) x_{\alpha_1 + \delta}(c_2) s_1 s_0 I \mid c_1, c_2 \in \mathbb{C}\}$$


since $I s_1 s_0 I \subseteq U_1^s s_0 I$.

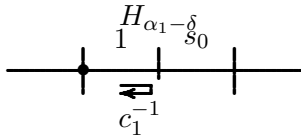
$I s_0 I = \{x_{-\alpha_1 + \delta}(c_1) s_0 I \mid c_1 \in \mathbb{C}\}$. *Case $c_1 \neq 0$:* Then

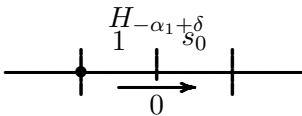
$$\begin{aligned} x_{-\alpha_1 + \delta}(c_1) s_0 I &= x_{-\alpha_1 + \delta}(c_1) x_{-\alpha_1 + \delta}(-c_1) x_{\alpha_1 - \delta}(c_1^{-1}) x_{-\alpha_1 + \delta}(-c_1) I \\ &= x_{\alpha_1 - \delta}(c_1^{-1}) x_{-\alpha_1 + \delta}(-c_1) I = x_{\alpha_1 - \delta}(c_1^{-1}) I. \end{aligned}$$

This is the case when a positive crossing changes to a fold.

Case $c_1 = 0$: Then $x_{-\alpha_1 + \delta}(0) s_0 I = s_0 I$.

So

$$\begin{aligned} I s_0 I \cap U^+ I &= \{x_{\alpha_1 - \delta}(c_1^{-1}) I \mid c_1 \neq 0\} \\ &= \{x_{-\alpha_1 + \delta}(c_1) I \mid c_1 \neq 0\}, \end{aligned}$$


$$I s_0 I \cap U^+ s_0 I = \{s_0 I\} = \{x_{-\alpha_1 + \delta}(0) s_0 I\}.$$


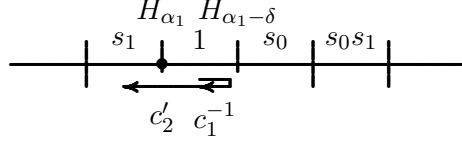
This is the case when a positive crossing changes to a negative crossing.

$$Is_1s_0I = \{x_{-\alpha_1+\delta}(c_1)x_{-\alpha_1+2\delta}(c_2)s_0s_1I \mid c_1c_2 \in \mathbb{C}\} = \{x_{-\alpha_1+\delta}(c_1)s_0x_{-\alpha_1}(c_2)s_1I \mid c_1, c_2 \in \mathbb{C}\}.$$

Case $c_1 \neq 0$: Then

$$\begin{aligned} x_{-\alpha_1+\delta}(c_1)s_0x_{-\alpha_1}(c_2)s_1I &= x_{-\alpha_1+\delta}(c_1)x_{-\alpha_1+\delta}(-c_1)x_{\alpha_1-\delta}(c_1^{-1})x_{-\alpha_1+\delta}(-c_1)x_{\alpha_1}(c_2)s_1I \\ &= x_{\alpha_1-\delta}(c_1^{-1})x_{-\alpha_1+\delta}(-c_1)x_{\alpha_1}(c_2)s_1I \\ &= x_{\alpha_1-\delta}(c_1^{-1})x_{-\alpha_1+\delta}(-c_1)Is_1I \\ &= x_{\alpha_1-\delta}(c_1^{-1})x_{\alpha_1}(c_2')s_1I \end{aligned}$$

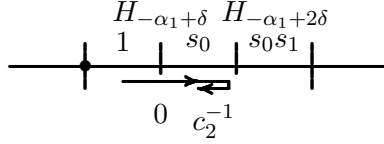
So



Case $c_1 = 0$ and $c_2 \neq 0$: Then

$$\begin{aligned} x_{-\alpha_1+\delta}(0)s_0x_{\alpha_1}(c_2)s_1I &= x_{-\alpha_1+\delta}(0)s_0x_{\alpha_1}(c_2)x_{\alpha_1}(-c_2)x_{-\alpha_1}(c_2^{-1})x_{\alpha_1}(-c_2)I \\ &= x_{-\alpha_1+\delta}(0)s_0x_{-\alpha_1}(c_2^{-1})x_{\alpha_1}(-c_2)I \\ &= x_{-\alpha_1+\delta}(0)s_0x_{-\alpha_1}(c_2^{-1})I \\ &= x_{-\alpha_1+\delta}(0)x_{\alpha_1+2\delta}(c_2^{-1})s_0I. \end{aligned}$$

So



Case $c_1 = 0$ and $c_2 = 0$: Then

$$x_{-\alpha_1+\delta}(0)s_0x_{\alpha_1}(0)s_1I = s_0s_1I.$$

A horizontal number line with tick marks at $H_{-\alpha_1+\delta}$, 1 , s_0 , $H_{-\alpha_1+2\delta}$, and s_0s_1 . Below the line, two right-pointing arrows are labeled 0 and 0 .