Hints and Solutions to Problems on the August 09 Qualifying Exam in Analysis [1]

Problem 1. *Let $b \geq 1$. A sequence $\{a_n\}_{n=0}^{\infty}$ of positive real numbers is defined inductively by specifying $a_0 > 0$, and then setting*

$$a_{n+1} = \frac{a_n}{2} + \frac{b^3}{2a_n^2}$$

for $n \geq 0$.

(a) Show that if $L = \lim_{n\to\infty} a_n$ exists, then $L = b$.
(b) Show that there is an open interval $I_b \subset \mathbb{R}$ containing b so if $a_0 \in I_b$, then $L = \lim_{n\to\infty} a_n$ exists.
(c) What can you say about the length of I_b?

Although it is not necessary to do so, one can easily reduce to the case $b = 1$, by setting $\widetilde{a}_n = ba_n$ for which the recurrence relation is $\widetilde{a}'_{n+1} = \frac{\widetilde{a}'_n}{2} + \frac{1}{2\widetilde{a}_n^2}$.

Therefore in (a) and (b), we set $b = 1$ and take henceforth

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n^2}.$$

(a) One has $a_{n+1}a_n^2 = \frac{a_n^3}{2} + \frac{1}{2}$. If $a_n \to L$, as $n \to +\infty$, taking limits, one gets $L^3 = \frac{L^3}{2} + \frac{1}{2}$. So $L^3 = 1$ and $L = 1$. (Without the above rescaling the answer is $L = b$). Note that the above writing avoids to have to discuss whether $L = 0$.

(b) For $x > 0$, set $g(x) = \frac{x}{2} + \frac{1}{2x^2}$. So $a_{n+1} = g(a_n)$. We have $g(1) = 1$, 1 is a fixed point of g, and $g'(x) = \frac{1}{2} - x^{-3}$ thus $g'(1) = -1/2$. It is easy to find an interval $(1 - h\ ,\ 1 + h)$, with $h < 1$, on which (say) $|g'(x)| < \frac{3}{4} < 1$ $(1 - h = (\frac{4}{5})^{1/3})$.

If $|1 - a_n| < h$ then $|1 - a_{n+1}| < \frac{3}{4}|1 - a_n| < h$. So, if $|1 - a_0| < h$, $|1 - a_n| < (\frac{3}{4})^n|1 - a_0|$, and $a_n \to 1$ as $n \to \infty$. (This is a standard contraction principle/shrinking lemma argument).

(c) By rescaling (or by directly working in the above argument with the original definition), one can take I_b with length bh, i.e. proportional to b.

Remark: Although the above was considered to be a satisfactory answer, one can show more. For any $a_0 > 0$ the sequence (a_n) converges to b. Graphing g is very helpful for understanding iteration of the map g. Note that there is never monotone convergence to b, unless $a_0 = b$.

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[1] Prepared by the exam committee.

Problem 2. *Let* $f(x) = \sum_{n=1}^{\infty} (1 + n^4 x^2)^{-1}$.

(a) *Show that* f *is continuously differentiable on* $(0, \infty)$.

This is a standard calculus problem. You can check that for any fixed $a > 0$, that the series $\sum f_n'$ and $\sum f_n$ converge uniformly on $[a, \infty)$.

(b) *Show that there is a constant* $C > 0$ *so that* $f(x) \le C\, x^{-\frac{1}{2}}$ *for all* $0 < x \le 1$, *and* $f(x) \le C\, x^{-2}$ *for* $x \ge 1$.

Assume $0 < x \le 1$. Note that for $n^4 x^2 \le 1$ the expression $(1 + n^4 x^2)^{-1}$ is comparable to 1 and for $n^4 x^2 \ge 1$ the bound $(1 + n^4 x^2)^{-1} \le n^{-4} x^{-2}$ is efficient.

For all $x > 0$, $(1 + n^4 x^2)^{-1} \le x^{-2} n^{-4}$ and thus $\sum_{n=1}^{\infty} (1 + n^4 x^2)^{-1} \le x^{-2} \sum_{n=1}^{\infty} n^{-4} \le C_1 x^{-2}$.

For $0 < x \le 1$ we can also estimate

$$\begin{aligned} |f(x)| &\le \sum_{n: n^4 \le x^{-2}} 1 + \sum_{n: n^4 > x^{-2}} (n^4 x^2)^{-1} \le x^{-1/2} + x^{-2} \sum_{n: n > x^{-1/2}} n^{-4} \\ &\le x^{-1/2} + x^{-2} \int_{t > x^{-1/2} - 1}^{\infty} t^{-4} dt \le C_2 x^{-1/2}. \end{aligned}$$

(c) *Show that the improper Riemann integral* $\int_0^{\infty} f(x)\, dx = \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \int_{\epsilon}^{N} f(x)\, dx$ *exists.*

Let $0 < \epsilon' < \epsilon < 1$, $1 < R < R'$. You may deduce the assertion from

$$\begin{aligned} \Big| \int_R^{R'} f(x) dx \Big| &\le \int_R^{R'} C_1 x^{-2} dx \le C_1 R^{-1} \\ \Big| \int_{\epsilon'}^{\epsilon} f(x) dx \Big| &\le \int_{\epsilon'}^{\epsilon} C_2 x^{-1/2} dx \le 2 C_2 \epsilon^{1/2}. \end{aligned}$$

Problem 3

Let X and Y be normed vector spaces with norms $||\cdot||_X$ and $||\cdot||_Y$. Let $\Omega \subset X$ be an open set and let $x \in \Omega$. Recall that a function $F : \Omega \to Y$ is differentiable at $x \in \Omega$ if there is a continuous linear transformation $S_x : X \to Y$ such that

$$\lim_{||h||_X \to 0} \frac{||F(x+h) - F(x) - S_x(h)||_Y}{||h||_X} = 0.$$

We then say that S_x is the derivative of F at x.

(a) *Let X, Y, and Z be normed vector spaces with norms $||\cdot||_X$, $||\cdot||_Y$, and $||\cdot||_Z$. Let $F : X \to Y$ be differentiable at a point $x_0 \in X$ with derivative S_{x_0}, and let $G : Y \to Z$ be differentiable at the point $F(x_0)$ with derivative $T_{F(x_0)}$. Prove that the composition $G \circ F : X \to Z$ defined by $G \circ F(x) = G\big(F(x)\big)$ is differentiable at x_0, and compute its derivative.*

This is the chain rule and the proof (in the general context needed here) can be found in many standard textbooks. For a nice and concise review of some basic facts in calculus we recommend chapter 1 of vol. 1 of "The analysis of linear partial differential operators", by L. Hörmander.

(b) *Let M_n denote the space of all real $n \times n$ matrices m, and define $F : M_n \to M_n$ by $F(m) = m^3$. Prove that F is differentiable at every matrix $m \in M_n$, and compute the derivative of F.*

Specify a norm on M_n (it does not matter which one since all are equivalent). It is convenient to work with the norm of m as a linear operator on the Euclidean space $\mathbb{C}^d$ (although not essential). This norm also satisfies the multiplication inequality $\|mm'\| \leq \|m\|\|m'\|$.

Expand for $m \in M_n$, $h \in M_n$

$$(m+h)^3 = m^3 + m^2h + mhm + hm^2 + mh^2 + hmh + h^2m + h^3$$

so that

$$(m+h)^3 - m^3 = m^2h + mhm + hm^2 + r(m,h)$$

where $\|r(m,h)\| \leq 3\|m\|\|h\|^2 + \|h\|^3$, hence $\lim_{\|h\|\to 0} \frac{\|r(m,h)\|}{\|h\|} = 0$. Thus the derivative is the linear operator DF_m defined by

$$h \mapsto m^2h + mhm + hm^2.$$

(Note that because of non-commutativity, this is usually *not* equal to $3m^2h$.)

(c) *Let $\Omega \subset M_n$ denote the set of invertible $n \times n$ matrices, and define $G : \Omega \to M_n$ by setting $G(m) = m^{-1}$. Prove that Ω is an open subset of M_n, and that G is differentiable at every point $m \in \Omega$, and compute the derivative of G.*

Let $m \in \Omega$ and let h be a matrix with small norm. Then $(m+h) = m(I + m^{-1}h)$. The matrix $(I + m^{-1}h)$ is invertible if $\|h\| < \|m^{-1}\|^{-1}$; then $(I + m^{-1}h)^{-1} = \sum_{n=0}^{\infty}(-1)^n(m^{-1}h)^n$. The inverse of $m+h$ is given by $(I + m^{-1}h)^{-1}m^{-1}$.

Now for $\|h\| < \|m^{-1}\|^{-1}$ we use the above sum to expand

$$(m+h)^{-1} = m^{-1} - m^{-1}hm^{-1} + r_2(m,h)$$

where $\lim_{h\to 0} \frac{r_2(m,h)}{\|h\|} = 0$ and the derivative is the linear operator $h \mapsto m^{-1}hm^{-1}$.

Problem 4

Let $\{f_0, f_1, \ldots, f_n, \ldots\}$ *be a sequence of Lebesgue measurable functions on the interval* $[0,1]$.

(a) *Suppose that*
(i) $\lim_{n\to\infty} f_n(x) = f_0(x)$ *for almost every* $x \in [0,1]$;
item $\int_0^1 |f_n(x)|\,dx < +\infty$ *for every* $n \geq 0$;
(ii) $\lim_{n\to\infty} \int_0^1 |f_n(x)|\,dx = \int_0^1 |f_0(x)|\,dx$.
Prove that $\displaystyle\lim_{n\to\infty} \int_0^1 |f_n(x) - f_0(x)|\,dx = 0$.

`Proof I` using Egorov's theorem:

Let $\epsilon > 0$. By Egorov's theorem, there is a set $E \subset [0,1]$, whose complement in $[0,1]$ will be denoted by E^c, such that f_n converges uniformly to f_0 on E, and $\int_{E^c} |f_0| \leq \epsilon$, (for this inequality to hold it is enough that the measure of E^c be small enough). Trivially $\int_E |f_n| \to \int_E |f_0|$. Since $\int_{[0,1]} |f_n| \to \int_{[0,1]} |f_0|$, by difference we get $\int_{E^c} |f_n| \to \int_{E^c} |f_0| \leq \epsilon$. For n large enough, $\int_{[0,1]} |f_0 - f_n| \leq \int_E |f_0 - f_n| + \int_{E^c} |f_0| + \int_{E^c} |f_n| \leq 4\epsilon$.

`Proof II`: This is a short but tricky proof using the Lebesgue dominated convergence theorem. We have $-|f_0(x)| \leq |f_n(x)| - |f_n(x) - f_0(x)| \leq |f_0(x)|$, so $\big||f_n(x)| - |f_n(x) - f_0(x)|\big| \leq |f_0(x)|$. Since $|f_n(x)| - |f_n(x) - f_0(x)| \to |f_0(x)|$ almost everywhere, it follows from the Lebesgue dominated convergence theorem. that

$$\int |f_0(x)|\,dx = \int \lim_{n\to\infty} \big[|f_n(x) - |f_n(x) - f_0(x)|\big]\,dx = \lim_{n\to\infty} \Big[\int |f_n(x)| - \int |f_n(x) - f_0(x)|\,dx\Big],$$

which, together with the hypothesis $\displaystyle\lim_{n\to\infty} \int |f_n(x)|\,dx = \int |f_0(x)|\,dx$, gives the desired conclusion.

(b) *Suppose that* $\displaystyle\lim_{n\to\infty} \int_0^1 |f_n(x) - f_0(x)|\,dx = 0$.
Prove that there is a subsequence $n_k \to \infty$ *such that* $\displaystyle\lim_{k\to\infty} f_{n_k}(x) = f_0(x)$ *for almost every* $x \in \mathbb{R}$.

Note that we are assuming that all f_n including f_0 are integrable. The key step consists in considering a subsequence (f_{n_k}) such that $\sum \|f_{n_{k+1}} - f_{n_k}\|_{L^1} < +\infty$. We shall take a subsequence (f_{n_k}) so that $\|f_{n_k} - f_{n_{k+1}}\|_{L^1} \leq 2^{-k}$. Very elementary considerations on measurable sets allow one to finish.

However, we indicate here another way to conclude by using "big tools":
Note that, by the monotone convergence Theorem, $G(x) = |f_{n_0}(x)| + \sum_k |f_{n_{k+1}}(x) - f_{n_k}(x)|$ (sum of a positive series, possibly ∞) defines an L^1 function (with norm $\leq \|f_{n_0}\|_{L^1} + \sum_{k=1}^{\infty} 2^{-k}$), that is therefore a.e. finite. The series $f_{n_0} + \sum_k (f_{n_{k+1}} - f_{n_k})$ is thus absolutely convergent to a function S a.e. , i.e. f_{n_k} tends a.e. to S. The Lebesgue dominated convergence theorem, with majorizing function G, implies that $f_{n_k} \to S$ in L^1. So $S = f_0$ a.e. .

Problem 5.

If $f \in L^1(\mathbb{R})$ *and* $y > 0$, *define* $f_y(x) = \dfrac{1}{\sqrt{y}} \displaystyle\int_{\mathbb{R}} f(x-t) e^{-\frac{\pi t^2}{y}}\, dt.$

(a) *Prove that for each* $y > 0$, *the function* $f_y \in L^1(\mathbb{R})$; *i.e.* $\displaystyle\int_{\mathbb{R}} |f_y(t)|\, dt < +\infty$ *for each* $y > 0$.

Use Fubini.

(b) *Prove that* $\displaystyle\lim_{y\to 0} \int_{\mathbb{R}} |f(x) - f_y(x)|\, dx = 0.$

Note that $\frac{1}{\sqrt{y}} \int_{\mathbb{R}} e^{-\frac{\pi t^2}{y}}\, dt = 1$ for all $y > 0$. Thus $f_y(x) - f(x) = \dfrac{1}{\sqrt{y}} \displaystyle\int_{\mathbb{R}} \big[f(x-t) - f(x)\big] e^{-\frac{\pi t^2}{y}}\, dt.$ We use the continuity of the translation on $L^1(\mathbb{R})$, i.e. $\lim_{t\to 0} \|f(\cdot - t) - f(\cdot)\|_{L^1} = 0$ (this in itself is a standard exercise).

Thus, given $\epsilon > 0$ there is a $\delta > 0$ so that $\|f(\cdot - t) - f(\cdot)\|_{L^1} < \epsilon/2$ for $|y| \le \delta$. Moreover for *fixed* δ we have

$$\int_{|t|>\delta} \frac{1}{\sqrt{y}} e^{-\frac{\pi t^2}{y}}\, dt = \int_{|s|>\delta y^{-1/2}} e^{-\pi s^2}\, ds \to 0 \quad \text{if } y \to 0.$$

Observe that

$$\|f_y - f\|_{L^1} \le \int_{|t|\le\delta} \frac{1}{\sqrt{y}} e^{-\frac{\pi t^2}{y}} \|f(\cdot - t) - f(\cdot)\|_{L^1} dt + 2\|f\|_{L^1} \int_{|t|>\delta} \frac{1}{\sqrt{y}} e^{-\frac{\pi t^2}{y}}\, dt.$$

Thus if $\epsilon > 0$ is given choose δ so that $\|f(\cdot - t) - f(\cdot)\|_{L^1} \le \epsilon/2$ for $|t| \le \delta$ and then choose y_0 so that $\int_{|t|>\delta} \frac{1}{\sqrt{y}} e^{-\frac{\pi t^2}{y}}\, dt \le \epsilon/(2\|f\|_{L^1})$ for $y < y_0$.

(c) *There exists a constant* $C > 0$ *such that for every* $f \in L^1(\mathbb{R})$,

$$\left|\left\{ x \in \mathbb{R} \,\Big|\, \sup_{y>0} |f_y(x)| > \lambda \right\}\right| \le \frac{C}{\lambda} \int_{\mathbb{R}} |f(t)|\, dt. \qquad (*)$$

Using this inequality, prove that if $f \in L^1(\mathbb{R})$, *then for almost every* $x \in \mathbb{R}$, $\displaystyle\lim_{y\to 0} f_y(x) = f(x)$. Let

$$E_n = \{x \in \mathbb{R} : \limsup_{y\to 0} |f_y(x) - f(x)| > \tfrac{1}{n}\}.$$

The assertion follows if we can show that $\cup_n E_n$ has measure zero, which follows if each E_n has measure zero. This in turn follows if for every $\epsilon > 0$ we can show that E_n has measure $\le \epsilon$. Let g be continuous function which is compactly supported, and which satisfies $\|g - f\|_{L^1} < c(n, f, g, \epsilon)$ small (to be determined, independently of y). Prove that $g - g_y$ converges uniformly to 0 as $\epsilon \to 0$ (the proof in (b) works if you take the L^∞ norm in place of the L^1 norm). Then we obtain $\limsup_{y\to 0} |f_y(x) - f(x)| = \limsup_{y\to 0} |(f-g)_y - (f-g)|$. The set E_n is contained in the union of sets $E_{n,1}$ and $E_{n,2}$ where $E_{n,1} = \{x : \sup_{y>0} |(f-g)_y(x)| > \frac{1}{2n}\}$, and where $E_{n,2} = \{x : |(f-g)(x)| > \frac{1}{2n}\}$. Clearly $|E_{n,2}| \le 2n\|f - g\|_{L^1}$, by Chebyshev's inequality, and $|E_{n,1}| \le C(2n)\|f - g\|_{L^1}$, by our hypothesis. Thus if above we use $c(n, f, g, \epsilon) \le (2n)^{-1}(1 + C)^{-1}\epsilon$ we deduce $|E| \le \epsilon$.

Additional exercise: Prove that (*), which was given, follows from the analogous inequality for the Hardy-Littlewood function of f.

Problem 6

(a) *For which real numbers* $a \in \mathbb{R}$ *and* $b > 0$ *is it true that* $\left|\int_0^N e^{ix^b}(1+x)^a\,dx\right|$ *is bounded independently of the number* $N > 0$*?*

(b) *For which real numbers* $a \in \mathbb{R}$ *and* $b > 0$ *is it true that the improper integral* $\left|\int_0^\infty e^{ix^b}(1+x)^a\,dx\right|$ *converges?*

To treat this problem, there was a possibility to separate real part and imaginary part, and to restrict to the consideration of the case $N^p = K\pi$ $(K \in \mathbb{N})$ *for the imaginary part, and* $N^p = \frac{\pi}{2} + K\pi$ *for the real part. The problem then reduced to a simple problem on alternate series. Here we present a proof based on integration by parts.*

1) If $b > a+1$, $\int_0^N e^{ix^b}(1+x)^a\ dx$ has a limit as $N \to \infty$.

Sketch of proof: Integration by parts, on the interval $[1, N]$ (one needs to avoid 0), yields

$$\begin{aligned}\frac{1}{ib}\int_1^N e^{ix^b}(1+x)^a\ dx &= \int_1^N (ibx^{b-1}e^{ix^b})\frac{(1+x)^a}{x^{b-1}}\\ &= -\int_1^N e^{ix^b}\left(\frac{(1+x)^a}{x^{b-1}}\right)' + \left[e^{ix^b}\frac{(1+x)^a}{x^{b-1}}\right]_1^N\ .\ (*)\end{aligned}$$

If $b = a+1+\epsilon$, $\epsilon > 0$, $|(\frac{(1+x)^a}{x^{b-1}})'| = O(\frac{1}{x^{1+\epsilon}})$, so $e^{ix^b}(\frac{(1+x)^a}{x^{b-1}})'$ is integrable on $(1,\infty)$, and $\frac{(1+x)^a}{x^{b-1}}$ tends to 0 at ∞.

2) If $b < a+1$ the integral $\int_0^N e^{ix^b}(1+x)^a\ dx$ is not bounded uniformly (i.e. independently of N).

Sketch of proof. The result is obtained by brutal estimates. Indeed, for $K \in \mathbb{N}$, consider the interval defined by $2K\pi - \frac{\pi}{4} < x^b < 2K\pi + \frac{\pi}{4}$. On that interval, whose length is $\geq \frac{C}{K^{\frac{b-1}{b}}}$ for some constant $C > 0$ (that depends on b), $\operatorname{Re} e^{ix^p} \geq \frac{1}{\sqrt{2}}$, so $\operatorname{Re}\left(e^{ix^b}(1+x)^a\right) \geq \frac{1}{\sqrt{2}}K^{\frac{a}{b}}$. Set $N_K = (2K\pi - \frac{\pi}{4})^{\frac{1}{b}}$ and $N'_K = (2K\pi + \frac{\pi}{4})^{\frac{1}{b}}$, and let $K \to +\infty$. One has

$$|\int_0^{N_K} e^{ix^b}(1+x)^a\ dx - \int_0^{N'_K} e^{ix^b}(1+x)^a\ dx| \geq \frac{C}{\sqrt{2}}\frac{K^{\frac{a}{b}}}{K^{\frac{b-1}{b}}} \to \infty\ ,$$

since $a > b-1$.

One can also get the result by more integrations by parts.

3) If $b = a+1$, the integral $\int_0^N e^{ix^b}(1+x)^a\ dx$ is bounded independently of N but there is no limit as $N \to \infty$. Simply look at what happens in 1) above if $b = a+1$. In $(*)$ the integral converges (decay in $\frac{1}{x^2}$ if $a \neq 0$, and 0 in the trivial case $a = 0$), but the integrated term, that becomes $[e^{ix^b}(1+\frac{1}{x})^a]$, oscillates because of the factor e^{ix^b}.

Other solution: Before integrating by parts one can change variables, setting $x^b = t$*. Try it and see what you prefer. There is at least one advantage, step 2) becomes totally transparent.*

Problem 7R

(a) *Let H_1 and H_2 be Hilbert spaces, and let $T : H_1 \to H_2$ be a continuous linear operator. Give a precise definition of the adjoint operator T^*.*

Let $y \in H_2$. The map $x \mapsto < Tx, y >_{H_2}$, from H_1 into $\mathbb{R}$ ($\mathbb{C}$ in case on complex Hilbert spaces) defines a continuous linear form on H_1. By the Riesz representation theorem there is a unique $z \in H_1$ such that, for all $x \in H_1$, $< x, z >_{H_1} = < Tx, y >_{H_2}$. One sets $T^*y = z$. One can summarize:

$$< x, T^*y >_{H_1} = < Tx, y >_{H_2} \quad .$$

(b) *Let $(a, b) \subset \mathbb{R}$ be a (possibly infinite) open interval. If $f \in L^2(a, b)$, explain what it means that the distributional derivative f' is also in $L^2(a, b)$.*

There exists $g \in L^2(a, b)$ such that:
For any $\varphi \in \mathcal{C}_0^\infty(a, b)$ (smooth function with compact support)

$$\int_a^b g\varphi \ = \ -\int_a^b f\varphi'$$

(formula of integration by parts $\int f'\varphi = -\int f\varphi'$). Then $f' = g$.

(c) *Let $\mathbb{R}^+$ denote the positive real axis $[0, \infty)$. Let $H^1(\mathbb{R})$ (respectively $H^1(\mathbb{R}^+)$) be the space of real-valued functions $f \in L^2(\mathbb{R})$ (respectively $f \in L^2(\mathbb{R}^+)$) such that the distributional derivative f' is also in $L^2(\mathbb{R})$ (respectively $L^2(\mathbb{R}^+)$). Then $H^1(\mathbb{R})$ and $H^1(\mathbb{R}^+)$ are Hilbert spaces with inner product given by*

$$\langle f, g\rangle_{H^1(\mathbb{R})} = \int_{\mathbb{R}} f(x)g(x)\,dx + \int_{\mathbb{R}} f'(x)g'(x)\,dx;$$

$$\langle f, g\rangle_{H^1(\mathbb{R}^+)} = \int_{\mathbb{R}^+} f(x)g(x)\,dx + \int_{\mathbb{R}^+} f'(x)g'(x)\,dx.$$

Let $T : H^1(\mathbb{R}) \to H^1(\mathbb{R}^+)$ be the mapping given by restriction. Compute explicitly the adjoint operator T^*.

If $g \in H^1(0, +\infty)$, then $h = T^*g \in H^1(\mathbb{R})$ is defined by the property $\langle f, h\rangle = \langle Tf, g\rangle$, and since all functions involved are assumed to be real valued this means

$$\int_{-\infty}^{+\infty} fh + f'h'\,dx = \int_0^{+\infty} fg + f'g'\,dx. \qquad (*)$$

To determine h we test first against $\mathcal{C}_0^\infty$ functions f, with compact support in $(-\infty, 0)$ or with compact support in $(0, +\infty)$. One gets $h - h'' = 0$ on $(0, +\infty)$ and $h - h'' = g - g''$ on $(0, +\infty)$. Taking into account integrability conditions, one gets $h(x) = Ce^x$ for $x < 0$, and $h(x) = g + Ae^{-x}$ for $x > 0$. Continuity of h implies that $C = g(0) + A$. Rewrite $(*)$:

$$\int_{-\infty}^{0} fh + f'h'\,dx = \int_0^{+\infty} f(g - h) + f'(g' - h')\,dx \ .$$

Cancellation of the boundary terms in the integration by parts (no longer testing only against functions f vanishing near 0) leads to $C = -A$. Thus the answer for $T^*g = h$ is:

$$T^*g(x) = \begin{cases} g(x) - \frac{g(0)}{2}e^{-x} & \text{if } x > 0, \\ \frac{g(0)}{2}e^{x} & \text{if } x < 0. \end{cases}$$

Comment: There is some flexibility for writing down the above argument completely, and totally rigorously. Do we consider h'' and g'' to be functions defined in the classical sense on $(-\infty, 0)$ and $(0, +\infty)$, or distributions? Recall that general H^1 functions are continuous (dimension 1), they not differentiable in the classical sense, and their second derivatives are not functions. One can stay entirely in the classical setting of functions that are continuous on $\mathbb{R}$, and piecewise smooth, smooth on $(-\infty, 0]$ and on $[0, +\infty]$. Indeed since T^* is a continuous operator it is enough to study T^* on its dense subspace $\mathcal{C}_0^\infty$. Also by density $(*)$ need to be checked only for $f \in \mathcal{C}_0^\infty$. Of course, we do not have such a possibility of imposing smoothness of h, but we can still try to find h smooth on $(-\infty, 0]$ and on $[0, +\infty)$. The only risk would be to not find the solution if we were asking too much (e.g. if we had asked h to be smooth at 0).

Problem 8R

Let g be a positive decreasing function defined on $(0, +\infty)$. Show that the following are equivalent:

(a) There exists a distribution T on $\mathbb{R}$ such that $\langle T, \varphi \rangle = \int_0^\infty \varphi(x) g(x)\, dx$ for all test functions $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ which have compact support in the open interval $(0, \infty)$.
(b) There exists a non-negative integer $k \in \mathbb{N}$ and a constant $C > 0$ such that for all $x \in (0,1)$, $g(x) \le C\, x^{-k}$.

$(b) \Rightarrow (a)$. One can define T by:

$$T(\varphi) = \int_{[0,1]} \big(\varphi - T_p(\varphi) g\big) \ + \ \int_{[1,\infty)} \varphi g \ ,$$

for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, where $T_p(\varphi)$ is the Taylor polynomial for φ of degree p (take any $p \ge k$) at 0.

Another solution: On $(0, +\infty)$, one has $g = \frac{d^{k+1}}{dx^{k+1}} h$, with h bounded near 0. Extend h to a function $\tilde{h}$ on $\mathbb{R}$ by setting $\tilde{h} = 0$ on the negative axis. Define the extension of g to be the distribution defined by $\frac{d^{k+1}}{dx^{k+1}} \tilde{h}$. For this implication, the hypothesis that g is decreasing is not needed.

$(a) \Rightarrow (b)$. Here we shall use that g is decreasing, and the continuity property in the definition of distributions is essential. Thus there exist $K \in \mathbb{N}$, $C \ge 0$ such that for all $\varphi \in \mathcal{C}_0^\infty$ with compact support in (say) $(0,1)$:

$$\Big| \int \varphi g \Big| = | < T, \varphi > | \le C \|\varphi\|_K \ ,$$

where $\|\varphi\|_{\mathcal{C}^K} = \max_{0 \le j \le K} \|\varphi^{(j)}\|_\infty$.

Let $\varphi \in \mathcal{C}_0^\infty$ be any non-negative function, not identically 0, with compact support in $(0,1)$; we may require the convenient normalization $\int \varphi = 1$. For any $0 < t < 1$, set $\varphi_t(x) = t^{-1} \varphi(x/t)$. The function φ_t has compact support in $(0,t)$. Since g is decreasing, and $\int \varphi_t = 1$, one has

$$\int \varphi_t(x) g(x) dx \ge g(t) \ .$$

Since $\|\varphi_t\|_{\mathcal{C}^K} \le \|\varphi\|_{\mathcal{C}^K} t^{-K-1}$ for $0 < t < 1$, we get $g(t) \le AC t^{-K-1}$ for all $t \in [0,1]$, with $A := \|\varphi\|_{\mathcal{C}^K}$.

Problem 9R

If $I \subset \mathbb{R}$ is a closed interval, a function $f : I \to \mathbb{R}$ is monotonic on I if either it is non-decreasing on I or it is non-increasing on I. A function f on the interval $[0,1]$ is said to be nowhere monotonic if there is no closed subinterval $I \subset [0,1]$ on which f is monotonic. Prove that there exists a continuous function f on $[0,1]$ which is nowhere monotonic.

Hint: It may be easier to show more: the "typical" function in the space $C[0,1]$ of real-valued continuous functions on $[0,1]$ is nowhere monotonic.

The hint indicates of course that the Baire category theorem can be used. There is a possible difficulty in that the number of open intervals is uncountable. However let's just consider the set of all closed subintervals of I with *rational* endpoints and enumerate these intervals by $J_1, J_2, \dots$. It is immediate that f is nowhere monotonic on I if and only if f is not monotonic on any of the intervals J_k, $k = 1, 2, \dots$.

Consider the set E_k^+ of all functions in $C(I)$ which are nondecreasing on J_k and the set E_k^- of all functions in $C(I)$ which are nonincreasing on J_k. Clearly both sets are closed in $C(I)$.

A function $f \in C(I)$ is nowhere monotonic if it is in the complement of $\bigcup_{k=1}^{\infty}(E_k^+ \cup E_k^-)$. Since $C(I)$ is complete the Baire category theorem tells us that this complement is not empty (indeed of second category) if we can show that none of the sets E_k^+ and E_k^- contains an interior point (i.e. they are nowhere dense since we have seen that they are closed).

Fix k and $g \in E_k^+$. Given $\epsilon > 0$ it is possible to find a function h with $\sup |h| < \epsilon$ so that $g + h$ is not nondecreasing on J_k (carry this out in detail!). Similar argument for E_k^-. Thus the $E_k^\pm$ are nowhere dense. Thus $\bigcup_{k=1}^{\infty}(E_k^+ \cup E_k^-)$ is of first category.

Problem 7C

Using contour integration, evaluate the convergent improper integral

$$\int_0^\infty \frac{\sqrt{x}\,\cos(x)}{1+x^2}\,dx.$$

Note: There is a problem here. Unfortunately, the examination committee was not careful, and this integral cannot be computed using the standard contour integral technique. If students applied the method correctly or essentially correctly, they received full or high credit for the problem.

Let's outline the standard technique and try to apply it to the problem at hand.

We shall take the branch of square root with $\sqrt{re^{i\alpha}} = r^{1/2}e^{i\alpha/2}$ for $-\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$. It is natural to consider the function $F(z) = \sqrt{z}e^{iz}(1+z^2)^{-1}$. F is analytic in the set of all $\{z : \mathrm{Im}(z) > 0\} \setminus \{i\}$ and $\lim_{y\to 0+} F(x+iy) = F(x)$ (in fact F is analytic away from the negative imaginary axis and away from $\{i\}$).

Let C_R denote the usual closed semicircle consisting of $I_R = [-R, R]$ and $S_R := \{z : z = Re^{it}, 0 \le t \le \pi\}$, with the counterclockwise orientation. Then the integral over C_R equals $2\pi i$ times the residue of F at i. Thus, for $R > 1$,

$$\int_{C_R} F(z)dz = 2\pi i \frac{e^{iz}\sqrt{z}}{2z}\Big|_{z=i} = \pi e^{-1}e^{i\pi/4}.$$

Also notice that $\lim_{R\to\infty}\int_{S_R} F(z)dz = 0$. This follows from the estimate $|e^{i(x+iy)}| \le 1$ for $y \ge 0$ and thus

$$|F(z)| \le \frac{\sqrt{R}}{R^2-1}, \text{ if } |z| = R \text{ and } \mathrm{Im}(z) \ge 0.$$

Hence

$$\Big|\int_{S_R} F(z)dz\Big| \le \mathrm{length}(S_R)\frac{\sqrt{R}}{R^2-1} \le \frac{\pi R^{3/2}}{R^2-1}$$

which tends to 0 as $R \to \infty$. Thus we obtain that

$$\lim_{R\to\infty}\int_{I_R} F(z)dz = \lim_{R\to\infty}\int_{C_R} F(z)dz = \pi e^{-1}e^{i\pi/4}.$$

We now attempt to relate this integral to the given one. Note that

$$\begin{aligned}\int_{I_R} F(z)dz &= \int_{-R}^0 i\sqrt{-x}e^{ix}(1+x^2)^{-1}dx + \int_0^R \sqrt{x}e^{ix}(1+x^2)^{-1}dx\\ &= \int_0^R \frac{\sqrt{x}}{1+x^2}(ie^{-ix}+e^{ix})dx\\ &= (1+i)\int_0^R \frac{\sqrt{x}}{1+x^2}(\cos x + \sin x)dx\end{aligned}$$

Thus if $A = \int_0^\infty \frac{\sqrt{x}}{1+x^2}\cos x\,dx$ and $B = \int_0^\infty \frac{\sqrt{x}}{1+x^2}\sin x\,dx$ we get

$$(1+i)(A+B) = \pi e^{-1}e^{i\pi/4}$$

This equation allows us to compute $A + B = \pi e^{-1}2^{-1/2}$, but unfortunately neither A nor B.

Remark. If $-1 < a < 1$ and $A' = \int_0^{+\infty}\frac{x^a\cos x}{1+x^2}\,dx$ and $B' = \int_0^{+\infty}\frac{x^a\sin x}{1+x^2}\,dz$, similar computations allow one to evaluate $(1+\cos(a\pi))A' + \sin(a\pi)B'$, but neither A' nor B'.

Problem 8C

Let $F : \mathbb{C} \to \mathbb{C}$ be an entire holomorphic function.

(a) *Prove that if* $\displaystyle\iint |F(x+iy)|^2\,dx\,dy < +\infty$*, then* $F(z) = 0$ *for all* $z \in \mathbb{C}$*.*

There are various ways to prove that $F = 0$ by using the mean value property. Here is another way, more in the spirit of Hilbert spaces and Fourier series.

Set $F(z) = \sum_{n=0}^{+\infty} a_n z^n$. One has

$$\int_0^{2\pi} |F(re^{i\theta})|^2 \, \frac{d\theta}{2\pi} \;=\; \sum_{n=0}^{+\infty} r^{2n} |a_n|^2 \;.$$

By hypothesis $\int_0^\infty \int_0^{2\pi} |F(re^{i\theta}|^2 \; d\theta \; rdr < \infty$. This clearly implies that $a_n = 0$ for all n.

(b) *Suppose that F is non-constant. Prove that for every $w \in \mathbb{C}$ and every $\epsilon > 0$ there exists $z \in \mathbb{C}$ so that $|F(z) - w| < \epsilon$. (**Note:** Do not just quote Picard's theorem.)*

By contradiction. Assume that there exists $w \in \mathbb{C}$ and $\epsilon > 0$, such that for all $z \in \mathbb{C}$, $|F(z) - w| > \epsilon$. Then set $G(z) = \frac{1}{F(z)-\epsilon}$, G is a bounded holomorphic function on $\mathbb{C}$. Hence G is constant and so is F.

(c) *Suppose that for each $w \in \mathbb{C}$, the equation $F(z) = w$ has at most 1 solution. Prove that there are complex numbers $a \neq 0$ and b so that $F(z) = az + b$.*

By subtraction of a constant one can assume that $F(0) = 0$. Since non constant holomorphic maps are open mappings, the image of the open unit disc is an open subset of $\mathbb{C}$. So, there exists $\delta > 0$, such that for all $\zeta \in \mathbb{C}$, with $|\zeta| < \delta$, there exists $z \in \mathbb{C}$ such that $|z| < 1$ and $F(z) = \zeta$. Then, injectivity of F implies that for $|z| > 1$ we have $|F(z)| > \delta$. Therefore $G(z) = \frac{1}{F(z)}$ defines a bounded holomorphic function on the complement of the unit disc, and ∞ is a removable singularity. One must have $G(\infty) = 0$, otherwise F would be bounded, hence constant. And G must have a simple zero at ∞ otherwise G hence F would not be injective. So, for some constant $C > 0$, and for all $|z|$ large, one must have $|G(z)| > \frac{C}{|z|}$, therefore $|F(z)| < \frac{1}{C}|z|$. So, since we assumed that $F(0) = 0$, $\frac{F(z)}{z}$ (extended by $F'(0)$ at 0) is a bounded holomorphic function on $\mathbb{C}$. Therefore $\frac{F(z)}{z}$ is constant, and $F(z) = az$, for some $a \neq 0$, and all $z \in \mathbb{C}$.

Remark: If one is not comfortable for dealing with functions defined at ∞, one can simply change variables and set $\tilde{G}(z) = G(\frac{1}{z})$. Then, 0 is a removable singularity for $\tilde{G}$.

Problem 9C

Let f be a function holomorphic in the open unit disk $\mathbb{D}$.

(a) *Suppose that $|f(z)| \leq 1$ for every $z \in \mathbb{D}$. Prove that $|f^{(N)}(z)| \leq N!(1-|z|)^{-N}$ for any integer $N \geq 1$ and any $z \in \mathbb{D}$.*

This is the very classical Cauchy estimate for derivatives. It follows immediately from the Cauchy formula for derivatives:

$$f^{(N)}(z) = \frac{N!}{2\pi i} \int_{|\zeta - z| = r} \frac{f(\zeta)}{(\zeta - z)^{N+1}} \, d\zeta \ .$$

Here, one has to take $r < 1 - |z|$, and let r tend to $(1 - |z|)$. One can also use rescaling.

(b) *Suppose that $|f(z)| \leq 1$ for every $z \in \mathbb{D}$. Let $0 \neq w_j \in \mathbb{D}$, and suppose that $f(w_j) = 0$ for $1 \leq j \leq N$. Prove that $|f(0)| \leq \prod_{j=1}^{N} |w_j|$. What can you conclude if there is equality?*

We suppose that $w_j \neq w_k$, if $j \neq k$.

A standard exercise on Blaschke products. Let

$$B(z) = \Pi_{j=1}^{N} \frac{z - w_j}{1 - \overline{w}_j z} \ .$$

Then B is a holomorphic function on a neighborhood of the closed unit disc, that has constant modulus 1 on the unit circle. Since B has simple zeroes at the points w_j and no other zeroes, there exists a holomorphic function g defined on the unit disc such that $f = Bg$. We have $|f| \leq 1$ and $\frac{|g(z)|}{|f(z)|}$ tends to 1 as $|z| \to 1$. By, the maximum principle, $|g| \leq 1$. So $|f(0)| \leq |B(0)| = \Pi |w_j|$. If equality holds, $f = e^{i\theta} B$, for some $\theta \in [0, 2\pi)$.

(c) *Let $\{x_n\}$ be a sequence of distinct real numbers such that $|x_n| < \frac{1}{2}$ for every $n \geq 1$, and suppose that $f(x_n)$ is real for every $n \geq 1$. Prove that $f(\bar{z}) = \overline{f(z)}$ for all $z \in \mathbb{D}$.*

Set $g(z) = \overline{f(\overline{z})}$, g is a holomorphic function on the unit disc. One has $f(x_n) = g(x_n)$, for all $n \geq 1$. By the isolated zeros Theorem, it follows that $f = g$. Taking complex conjugates, one gets $f(\overline{z}) = \overline{f(z)}$.