Bounded Stationary Reflection

Chris Lambie-Hanson

Department of Mathematical Sciences
Carnegie Mellon University

GSCL XV
Madison, WI
26 April 2014
Section 1

Jónsson cardinals
Jónsson algebras

Definition

• Let $X$ be a set and $\kappa$ a cardinal. $\mathbb{X}_\kappa = \{Y \subseteq X | |Y| = \kappa\}$.

• An algebra is a structure $\langle X, f_i | i < \omega \rangle$ such that, for every $i < \omega$, there is $n_i < \omega$ such that $f_i : \mathbb{X}_{n_i} \to X$.

• Equivalently, an algebra is a set $X$ together with a single function $f : \mathbb{X}_{<\omega} \to X$.

• If $\langle X, f \rangle$ is an algebra and $Y \subseteq X$, then $\langle Y, f \downharpoonright \mathbb{Y} \rangle_{<\omega}$ is a subalgebra of $\langle X, f \rangle$ if $f \downharpoonright \mathbb{Y} \subseteq Y$.

• An algebra $\langle X, f \rangle$ is a Jónsson algebra if it has no proper subalgebras of the same cardinality.
Jónsson algebras

Definition

- Let $X$ be a set and $\kappa$ a cardinal. $[X]^\kappa = \{ Y \subseteq X \mid |Y| = \kappa \}$.
Jónsson algebras

Definition

• Let $X$ be a set and $\kappa$ a cardinal. $[X]^{\kappa} = \{ Y \subseteq X \mid |Y| = \kappa \}$.
• An algebra is a structure $\langle X, f_i \mid i < \omega \rangle$ such that, for every $i < \omega$, there is $n_i < \omega$ such that $f_i : [X]^{n_i} \rightarrow X$.
• Similarly, an algebra $\langle X, f_i \rangle$ is a Jónsson algebra if it has no proper subalgebras of the same cardinality.
Jónsson algebras

Definition

• Let $X$ be a set and $\kappa$ a cardinal. $[X]^\kappa = \{ Y \subseteq X \mid |Y| = \kappa \}$.

• An algebra is a structure $\langle X, f_i \mid i < \omega \rangle$ such that, for every $i < \omega$, there is $n_i < \omega$ such that $f_i : [X]^{n_i} \rightarrow X$.

• Equivalently, an algebra is a set $X$ together with a single function $f : [X]^{<\omega} \rightarrow X$. 
Jónsson algebras

Definition

- Let $X$ be a set and $\kappa$ a cardinal. $[X]^\kappa = \{ Y \subseteq X \mid |Y| = \kappa \}$.
- An algebra is a structure $\langle X, f_i \mid i < \omega \rangle$ such that, for every $i < \omega$, there is $n_i < \omega$ such that $f_i : [X]^{n_i} \to X$.
- Equivalently, an algebra is a set $X$ together with a single function $f : [X]^{<\omega} \to X$.
- If $\langle X, f \rangle$ is an algebra and $Y \subseteq X$, then $\langle Y, f \mid [Y]^{<\omega} \rangle$ is a subalgebra of $\langle X, f \rangle$ if $f"[Y]^{<\omega} \subseteq Y$. 

Jónsson algebras

Definition

• Let $X$ be a set and $\kappa$ a cardinal. $[X]^\kappa = \{ Y \subseteq X \mid |Y| = \kappa \}$.
• An algebra is a structure $\langle X, f_i \mid i < \omega \rangle$ such that, for every $i < \omega$, there is $n_i < \omega$ such that $f_i : [X]^{n_i} \to X$.
• Equivalently, an algebra is a set $X$ together with a single function $f : [X]^{<\omega} \to X$.
• If $\langle X, f \rangle$ is an algebra and $Y \subseteq X$, then $\langle Y, f \upharpoonright [Y]^{<\omega} \rangle$ is a subalgebra of $\langle X, f \rangle$ if $f \upharpoonright [Y]^{<\omega} \subseteq Y$.
• An algebra $\langle X, f \rangle$ is a Jónsson algebra if it has no proper subalgebras of the same cardinality.
Jónsson cardinals

Definition

Let $\kappa$ be an infinite cardinal. $\kappa$ is a Jónsson cardinal if there are no Jónsson algebras $\langle X, f \rangle$ such that $|X| = \kappa$.

Proposition

Let $\kappa$ be an infinite cardinal. The following are equivalent.

1. $\kappa$ is a Jónsson cardinal.
2. For every $f : [\kappa]^\omega \rightarrow \kappa$, there is $H \in [\kappa][\kappa]$ such that $f"[H] < \omega$ is a proper subset of $\kappa$ (i.e. $\kappa \rightarrow [\kappa][\kappa]^\omega \kappa$).
3. For all sufficiently large, regular $\theta$ and all $x \in H(\theta)$, there is $M \prec H(\theta)$ such that:
   - $\{\kappa, x\} \in M$.
   - $|M \cap \kappa| = \kappa$.
   - $\kappa \not\subseteq M$. 

Jónsson cardinals

Definition
Let $\kappa$ be an infinite cardinal. $\kappa$ is a Jónsson cardinal if there are no Jónsson algebras $\langle X, f \rangle$ such that $|X| = \kappa$.

Proposition
Let $\kappa$ be an infinite cardinal. The following are equivalent.
1. $\kappa$ is a Jónsson cardinal.
2. For every $f : \left[ \kappa \right] \rightarrow \kappa$, there is $H \in \left[ \kappa \right] \kappa$ such that $f" [H] \subset < \omega$ (i.e. $\kappa \rightarrow \left[ \kappa \right] \rightarrow \kappa$).
3. For all sufficiently large, regular $\theta$ and all $x \in H(\theta)$, there is $M \prec H(\theta)$ such that:
   • $\{\kappa, x\} \in M$.
   • $|M \cap \kappa| = \kappa$.
   • $\kappa \not\subseteq M$. 
Jónsson cardinals

Definition
Let $\kappa$ be an infinite cardinal. $\kappa$ is a Jónsson cardinal if there are no Jónsson algebras $\langle X, f \rangle$ such that $|X| = \kappa$.

Proposition
Let $\kappa$ be an infinite cardinal. The following are equivalent.

1. $\kappa$ is a Jónsson cardinal.
2. For every $f: [\kappa] < \omega \rightarrow \kappa$, there is $H \in [\kappa]$ such that $f^{-1}([H]) < \omega$ is a proper subset of $\kappa$ (i.e. $\kappa \rightarrow [\kappa] < \omega \kappa$).
3. For all sufficiently large, regular $\theta$ and all $x \in H(\theta)$, there is $M \prec H(\theta)$ such that:
   - $\{\kappa, x\} \in M$.
   - $|M \cap \kappa| = \kappa$.
   - $\kappa \not\subseteq M$. 

Let $\kappa$ be an infinite cardinal. The following are equivalent.
Jónsson cardinals

Definition
Let $\kappa$ be an infinite cardinal. $\kappa$ is a Jónsson cardinal if there are no Jónsson algebras $\langle X, f \rangle$ such that $|X| = \kappa$.

Proposition
Let $\kappa$ be an infinite cardinal. The following are equivalent.

1. $\kappa$ is a Jónsson cardinal.

2. For every $f : [\kappa] < \omega \rightarrow \kappa$, there is $H \in [\kappa] \kappa$ such that $f" [H] < \omega$ is a proper subset of $\kappa$ (i.e. $\kappa \rightarrow [\kappa] < \omega \kappa$).

3. For all sufficiently large, regular $\theta$ and all $x \in H(\theta)$, there is $M \prec H(\theta)$ such that:
   • $\{\kappa, x\} \in M$.
   • $|M \cap \kappa| = \kappa$.
   • $\kappa \not\subseteq M$. 
Jónsson cardinals

Definition
Let $\kappa$ be an infinite cardinal. $\kappa$ is a Jónsson cardinal if there are no Jónsson algebras $\langle X, f \rangle$ such that $|X| = \kappa$.

Proposition
Let $\kappa$ be an infinite cardinal. The following are equivalent.

1. $\kappa$ is a Jónsson cardinal.
2. For every $f : [\kappa]^\omega \to \kappa$, there is $H \in [\kappa]^\kappa$ such that $f\,\{H\}^\omega$ is a proper subset of $\kappa$ (i.e. $\kappa \to [\kappa]^\kappa\omega$).
Jónsson cardinals

Definition
Let $\kappa$ be an infinite cardinal. $\kappa$ is a Jónsson cardinal if there are no Jónsson algebras $\langle X, f \rangle$ such that $|X| = \kappa$.

Proposition
Let $\kappa$ be an infinite cardinal. The following are equivalent.

1. $\kappa$ is a Jónsson cardinal.
2. For every $f : [\kappa]^{<\omega} \to \kappa$, there is $H \in [\kappa]^{\kappa}$ such that $f^{-1}[H]^{<\omega}$ is a proper subset of $\kappa$ (i.e. $\kappa \to [\kappa]^{<\omega}$).
3. For all sufficiently large, regular $\theta$ and all $x \in H(\theta)$, there is $M \prec H(\theta)$ such that:
   - $\{\kappa, x\} \in M$.
   - $|M \cap \kappa| = \kappa$.
   - $\kappa \notin M$. 
The large cardinal hierarchy

Proposition
Let $\kappa$ be a cardinal. Then $\kappa$ is measurable $\Rightarrow$ $\kappa$ is Ramsey $\Rightarrow$ $\kappa$ is Rowbottom $\Rightarrow$ $\kappa$ is Jónsson.

Theorem (Kleinberg)
Con(ZFC + there is a Rowbottom cardinal) $\iff$ Con(ZFC + there is a Jónsson cardinal)

Theorem (Prikry)
1 If $\kappa$ is a singular limit of measurable cardinals, then $\kappa$ is Jónsson.
2 If $\kappa$ is a measurable cardinal and $P$ is Prikry forcing at $\kappa$, then $\kappa$ remains Jónsson in $V_P$. 

The large cardinal hierarchy

Proposition

Let $\kappa$ be a cardinal. Then $\kappa$ is measurable $\Rightarrow$ $\kappa$ is Ramsey $\Rightarrow$ $\kappa$ is Rowbottom $\Rightarrow$ $\kappa$ is Jónsson.
The large cardinal hierarchy

Proposition

Let $\kappa$ be a cardinal. Then $\kappa$ is measurable $\Rightarrow$ $\kappa$ is Ramsey $\Rightarrow$ $\kappa$ is Rowbottom $\Rightarrow$ $\kappa$ is Jónsson.

Theorem (Kleinberg)

$\text{Con}(\text{ZFC} + \text{there is a Rowbottom cardinal}) \iff \text{Con}(\text{ZFC} + \text{there is a Jónsson cardinal})$
The large cardinal hierarchy

Proposition

Let \( \kappa \) be a cardinal. Then \( \kappa \) is measurable \( \Rightarrow \) \( \kappa \) is Ramsey \( \Rightarrow \) \( \kappa \) is Rowbottom \( \Rightarrow \) \( \kappa \) is Jónsson.

Theorem (Kleinberg)

\[ \text{Con}(\text{ZFC} + \text{there is a Rowbottom cardinal}) \iff \text{Con}(\text{ZFC} + \text{there is a Jónsson cardinal}) \]

Theorem (Prikry)

1. If \( \kappa \) is a singular limit of measurable cardinals, then \( \kappa \) is Jónsson.
The large cardinal hierarchy

Proposition
Let \( \kappa \) be a cardinal. Then \( \kappa \) is measurable \( \Rightarrow \) \( \kappa \) is Ramsey \( \Rightarrow \) \( \kappa \) is Rowbottom \( \Rightarrow \) \( \kappa \) is Jónsson.

Theorem (Kleinberg)
\( \text{Con}(\text{ZFC} + \text{there is a Rowbottom cardinal}) \Leftrightarrow \text{Con}(\text{ZFC} + \text{there is a Jónsson cardinal}) \)

Theorem (Prikry)
1. If \( \kappa \) is a singular limit of measurable cardinals, then \( \kappa \) is Jónsson.
2. If \( \kappa \) is a measurable cardinal and \( \mathbb{P} \) is Prikry forcing at \( \kappa \), then \( \kappa \) remains Jónsson in \( V^\mathbb{P} \).
Restrictions on Jónsson cardinals

Proposition

$\omega$ is not a Jónsson cardinal.
Restrictions on Jónsson cardinals

Proposition

\( \omega \) is not a Jónsson cardinal.

Proof.

Let \( f : [\omega]^{<\omega} \to \omega \) be such that, for every \( n < \omega \), \( f(\{n + 1\}) = n \).

Then \( \langle \omega, f \rangle \) is a Jónsson algebra. \( \square \)
Restrictions on Jónsson cardinals

Proposition

Let $\kappa$ be an infinite cardinal. If $\kappa$ is not a Jónsson cardinal, then $\kappa^+$ is not a Jónsson cardinal.
Restrictions on Jónsson cardinals

Proposition

Let $\kappa$ be an infinite cardinal. If $\kappa$ is not a Jónsson cardinal, then $\kappa^+$ is not a Jónsson cardinal.

Proof.

Suppose $\kappa$ is not a Jónsson cardinal. Then there is a regular $\theta > \kappa^+$ and $x \in H(\theta)$ such that, for all $M \prec H(\theta)$ with $\{\kappa, x\} \in M$ and $|M \cap \kappa| = \kappa$, then $\kappa \subseteq M$. 
Restrictions on Jónsson cardinals

Proposition

Let \( \kappa \) be an infinite cardinal. If \( \kappa \) is not a Jónsson cardinal, then \( \kappa^+ \) is not a Jónsson cardinal.

Proof.
Suppose \( \kappa \) is not a Jónsson cardinal. Then there is a regular \( \theta > \kappa^+ \) and \( x \in H(\theta) \) such that, for all \( M < H(\theta) \) with
\[ \{\kappa, x\} \in M \text{ and } |M \cap \kappa| = \kappa, \text{ then } \kappa \subseteq M. \]
Let \( M < H(\theta) \) be such that \( \{\kappa^+, x\} \in M \) and \( |M \cap \kappa^+| = \kappa^+ \). There is \( \alpha \in M \cap \kappa^+ \) such that \( |M \cap \alpha| = \kappa \). Since \( M \) contains a bijection between \( \kappa \) and \( \alpha \), \( |M \cap \kappa| = \kappa \). Thus, \( \kappa \subseteq M \).
Restrictions on Jónsson cardinals

Proposition

Let $\kappa$ be an infinite cardinal. If $\kappa$ is not a Jónsson cardinal, then $\kappa^+$ is not a Jónsson cardinal.

Proof.
Suppose $\kappa$ is not a Jónsson cardinal. Then there is a regular $\theta > \kappa^+$ and $x \in H(\theta)$ such that, for all $M \prec H(\theta)$ with \{$\kappa, x$\} $\in M$ and $|M \cap \kappa| = \kappa$, then $\kappa \subseteq M$. Let $M \prec H(\theta)$ be such that \{$\kappa^+, x$\} $\in M$ and $|M \cap \kappa^+| = \kappa^+$. There is $\alpha \in M \cap \kappa^+$ such that $|M \cap \alpha| = \kappa$. Since $M$ contains a bijection between $\kappa$ and $\alpha$, $|M \cap \kappa| = \kappa$. Thus, $\kappa \subseteq M$. For every $\beta \in M \cap \kappa^+$, since $M$ contains a bijection between $\kappa$ and $\beta$, we have $\beta \subseteq M$. Since $M \cap \kappa^+$ is unbounded in $\kappa^+$, we have $\kappa^+ \subseteq M$. $\square$
Restrictions on Jónsson cardinals

Theorem (Keisler-Rowbottom)

If $V = L$, then there are no Jónsson cardinals.
Restrictions on Jónsson cardinals

Theorem (Keisler-Rowbottom)
If $V = L$, then there are no Jónsson cardinals.

Theorem (Erdős-Hajnal-Rado)
If $2^\kappa = \kappa^+$, then $\kappa^+$ is not Jónsson.
Restrictions on Jónsson cardinals

Theorem (Keisler-Rowbottom)
If $V = L$, then there are no Jónsson cardinals.

Theorem (Erdős-Hajnal-Rado)
If $2^{\kappa} = \kappa^+$, then $\kappa^+$ is not Jónsson.

Theorem (Shelah)
If $\kappa$ is a singular cardinal and $\kappa$ is not a limit of regular Jónsson cardinals, then $\kappa^+$ is not Jónsson.
Stationary reflection

Definition
Let $\kappa$ be an uncountable, regular cardinal.

1. $S \subseteq \kappa$ is stationary in $\kappa$ if, for every closed, unbounded $C \subseteq \kappa$, $S \cap C \neq \emptyset$.
Stationary reflection

Definition
Let $\kappa$ be an uncountable, regular cardinal.

1. $S \subseteq \kappa$ is *stationary in $\kappa$* if, for every closed, unbounded $C \subseteq \kappa$, $S \cap C \neq \emptyset$.

2. If $S \subseteq \kappa$ is stationary and $\alpha < \kappa$, then $S$ reflects at $\alpha$ if $\text{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in $\alpha$.
Stationary reflection

Definition
Let $\kappa$ be an uncountable, regular cardinal.

1. $S \subseteq \kappa$ is *stationary in $\kappa$* if, for every closed, unbounded $C \subseteq \kappa$, $S \cap C \neq \emptyset$.

2. If $S \subseteq \kappa$ is stationary and $\alpha < \kappa$, then $S$ *reflects at $\alpha$* if $\text{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in $\alpha$. $S$ *reflects* if there is $\alpha < \kappa$ such that $S$ reflects at $\alpha$.

Theorem (Tryba, Woodin)
Suppose $\kappa$ is a regular Jónsson cardinal. Then every stationary subset of $\kappa$ reflects.

Remark
If $\kappa$ is a regular cardinal and $S \subseteq \kappa^+ \cap \text{cof}(\kappa)$, then $S$ cannot reflect. Thus, if $\kappa$ is regular, then $\kappa^+$ is not a Jónsson cardinal.
Stationary reflection

Definition
Let $\kappa$ be an uncountable, regular cardinal.

1. $S \subseteq \kappa$ is stationary in $\kappa$ if, for every closed, unbounded $C \subseteq \kappa$, $S \cap C \neq \emptyset$.

2. If $S \subseteq \kappa$ is stationary and $\alpha < \kappa$, then $S$ reflects at $\alpha$ if $\text{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in $\alpha$. $S$ reflects if there is $\alpha < \kappa$ such that $S$ reflects at $\alpha$.

Theorem (Tryba, Woodin)
Suppose $\kappa$ is a regular Jónsson cardinal. Then every stationary subset of $\kappa$ reflects.
Stationary reflection

Definition
Let $\kappa$ be an uncountable, regular cardinal.

1. $S \subseteq \kappa$ is stationary in $\kappa$ if, for every closed, unbounded $C \subseteq \kappa$, $S \cap C \neq \emptyset$.

2. If $S \subseteq \kappa$ is stationary and $\alpha < \kappa$, then $S$ reflects at $\alpha$ if $\text{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in $\alpha$. $S$ reflects if there is $\alpha < \kappa$ such that $S$ reflects at $\alpha$.

Theorem (Tryba, Woodin)
Suppose $\kappa$ is a regular Jónsson cardinal. Then every stationary subset of $\kappa$ reflects.

Remark
If $\kappa$ is a regular cardinal and $S \subseteq \kappa^+ \cap \text{cof}(\kappa)$, then $S$ cannot reflect. Thus, if $\kappa$ is regular, then $\kappa^+$ is not a Jónsson cardinal.
Stationary reflection

The proof of Tryba and Woodin’s theorem actually yields the following, apparently stronger statement:

**Fact**

Suppose \( \kappa \) is a regular Jonsson cardinal, \( S \subseteq \kappa \) is stationary, and \( \lambda < \kappa \) is a regular cardinal. Then there is \( \alpha \in \kappa \cap \text{cof}(\geq \lambda) \) such that \( S \) reflects at \( \alpha \).

In other words, every stationary subset of \( \kappa \) reflects at ordinals of arbitrarily high cofinality.

**Question (Eisworth)**

Suppose \( \mu \) is a singular cardinal, \( \kappa = \mu^+ \), and every stationary subset of \( \kappa \) reflects. Must it be the case that every stationary subset of \( \kappa \) reflects at ordinals of arbitrarily high cofinality?
Stationary reflection

The proof of Tryba and Woodin’s theorem actually yields the following, apparently stronger statement:

Fact

Suppose $\kappa$ is a regular Jónsson cardinal, $S \subseteq \kappa$ is stationary, and $\lambda < \kappa$ is a regular cardinal. Then there is $\alpha \in \kappa \cap \text{cof}(\geq \lambda)$ such that $S$ reflects at $\alpha$. 

Question (Eisworth)

Suppose $\mu$ is a singular cardinal, $\kappa = \mu^+$, and every stationary subset of $\kappa$ reflects. Must it be the case that every stationary subset of $\kappa$ reflects at ordinals of arbitrarily high cofinality?
Stationary reflection

The proof of Tryba and Woodin’s theorem actually yields the following, apparently stronger statement:

Fact

Suppose $\kappa$ is a regular Jónsson cardinal, $S \subseteq \kappa$ is stationary, and $\lambda < \kappa$ is a regular cardinal. Then there is $\alpha \in \kappa \cap \text{cof}(\geq \lambda)$ such that $S$ reflects at $\alpha$. In other words, every stationary subset of $\kappa$ reflects at ordinals of arbitrarily high cofinality.
Stationary reflection

The proof of Tryba and Woodin’s theorem actually yields the following, apparently stronger statement:

Fact
Suppose $\kappa$ is a regular Jónsson cardinal, $S \subseteq \kappa$ is stationary, and $\lambda < \kappa$ is a regular cardinal. Then there is $\alpha \in \kappa \cap \text{cof}(\geq \lambda)$ such that $S$ reflects at $\alpha$. In other words, every stationary subset of $\kappa$ reflects at ordinals of arbitrarily high cofinality.

Question (Eisworth)
Suppose $\mu$ is a singular cardinal, $\kappa = \mu^+$, and every stationary subset of $\kappa$ reflects. Must it be the case that every stationary subset of $\kappa$ reflects at ordinals of arbitrarily high cofinality?
Section 2

Approachability and forcing
Approachability

Definition

Let $\mu$ be a singular cardinal, and let $\kappa = \mu +$. Let $\vec{a} = \langle a_\alpha | \alpha < \kappa \rangle$ be a sequence of bounded subsets of $\kappa$. A limit ordinal $\beta < \kappa$ is approachable with respect to $\vec{a}$ if there is an unbounded $A \subseteq \beta$ of order type $\text{cf}(\beta)$ such that, for every $\gamma < \beta$, there is $\alpha < \beta$ such that $A \cap \gamma = a_\alpha$.

Let $S \subseteq \kappa$. $S \in I[\kappa]$ if there is a sequence $\vec{a}$ and a club $C \subseteq \kappa$ such that, for every $\beta \in S \cap C$, $\beta$ is approachable with respect to $\vec{a}$.

The approachability property holds at $\mu$ (written $\text{AP}_\mu$) if $\kappa \in I[\kappa]$. 
Approachability

Definition
Let $\mu$ be a singular cardinal, and let $\kappa = \mu^+$. 

Let $S \subseteq \kappa$. $S \in I[\kappa]$ if there is a sequence $\vec{a}$ and a club $C \subseteq \kappa$ such that, for every $\beta \in S \cap C$, $\beta$ is approachable with respect to $\vec{a}$.
Let $\mu$ be a singular cardinal, and let $\kappa = \mu^+$. 

1. Let $\bar{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ be a sequence of bounded subsets of $\kappa$. A limit ordinal $\beta < \kappa$ is *approachable with respect to* $\bar{a}$ if there is an unbounded $A \subseteq \beta$ of order type $\text{cf}(\beta)$ such that, for every $\gamma < \beta$, there is $\alpha < \beta$ such that $A \cap \gamma = a_\alpha$. 

The approachability property holds at $\mu$ (written $\text{AP}_\mu$) if $\kappa \in I[\kappa]$. 

---

**Approachability**

**Definition**

Let $\mu$ be a singular cardinal, and let $\kappa = \mu^+$. 

1. Let $\bar{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ be a sequence of bounded subsets of $\kappa$. A limit ordinal $\beta < \kappa$ is *approachable with respect to* $\bar{a}$ if there is an unbounded $A \subseteq \beta$ of order type $\text{cf}(\beta)$ such that, for every $\gamma < \beta$, there is $\alpha < \beta$ such that $A \cap \gamma = a_\alpha$. 

The approachability property holds at $\mu$ (written $\text{AP}_\mu$) if $\kappa \in I[\kappa]$. 

Approachability

Definition
Let $\mu$ be a singular cardinal, and let $\kappa = \mu^+$.  

1. Let $\vec{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ be a sequence of bounded subsets of $\kappa$. A limit ordinal $\beta < \kappa$ is approachable with respect to $\vec{a}$ if there is an unbounded $A \subseteq \beta$ of order type $\text{cf}(\beta)$ such that, for every $\gamma < \beta$, there is $\alpha < \beta$ such that $A \cap \gamma = a_\alpha$.

2. Let $S \subseteq \kappa$. $S \in I[\kappa]$ if there is a sequence $\vec{a}$ and a club $C \subseteq \kappa$ such that, for every $\beta \in S \cap C$, $\beta$ is approachable with respect to $\vec{a}$.

The approachability property holds at $\mu$ (written $\text{AP}_\mu$) if $\kappa \in I[\kappa]$. 
Approachability

Definition
Let $\mu$ be a singular cardinal, and let $\kappa = \mu^+$.

1. Let $\vec{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ be a sequence of bounded subsets of $\kappa$. A limit ordinal $\beta < \kappa$ is approachable with respect to $\vec{a}$ if there is an unbounded $A \subseteq \beta$ of order type $\text{cf}(\beta)$ such that, for every $\gamma < \beta$, there is $\alpha < \beta$ such that $A \cap \gamma = a_\alpha$.

2. Let $S \subseteq \kappa$. $S \in I[\kappa]$ if there is a sequence $\vec{a}$ and a club $C \subseteq \kappa$ such that, for every $\beta \in S \cap C$, $\beta$ is approachable with respect to $\vec{a}$.

3. The approachability property holds at $\mu$ (written $\text{AP}_\mu$) if $\kappa \in I[\kappa]$. 
Approachability

Definition
Let $\mu$ be a singular cardinal, $\kappa = \mu^+$, and $\theta$ a sufficiently large, regular cardinal. If $A$ is a countable expansion of $\langle H(\theta), \in, <_\theta \rangle$ and $\beta < \kappa$ is a limit ordinal, then $\beta$ is approachable with respect to $A$ if there is an unbounded $A \subseteq \beta$ of order type $\text{cf}(\beta)$ such that every initial segment of $A$ is in $\text{Sk}^A(\beta)$.
Approachability

Definition
Let \( \mu \) be a singular cardinal, \( \kappa = \mu^+ \), and \( \theta \) a sufficiently large, regular cardinal. If \( A \) is a countable expansion of \( \langle H(\theta), \in, <_\theta \rangle \) and \( \beta < \kappa \) is a limit ordinal, then \( \beta \) is approachable with respect to \( A \) if there is an unbounded \( A \subseteq \beta \) of order type \( \text{cf}(\beta) \) such that every initial segment of \( A \) is in \( Sk^A(\beta) \).

Fact
Let \( \mu \) be a singular cardinal, and let \( \kappa = \mu^+ \).
Approachability

Definition
Let $\mu$ be a singular cardinal, $\kappa = \mu^+$, and $\theta$ a sufficiently large, regular cardinal. If $\mathcal{A}$ is a countable expansion of $\langle H(\theta), \in, <_\theta \rangle$ and $\beta < \kappa$ is a limit ordinal, then $\beta$ is **approachable with respect to** $\mathcal{A}$ if there is an unbounded $\mathcal{A} \subseteq \beta$ of order type $\text{cf}(\beta)$ such that every initial segment of $\mathcal{A}$ is in $\text{Sk}^\mathcal{A}(\beta)$.

Fact
Let $\mu$ be a singular cardinal, and let $\kappa = \mu^+$.

1. $S \in \mathcal{I}[\kappa]$ if and only if there is a sufficiently large, regular $\theta$, a countable expansion $\mathcal{A}$ of $\langle H(\theta), \in, <_\theta \rangle$, and a club $C \subseteq \kappa$ such that, for every $\beta \in S \cap C$, $\beta$ is approachable with respect to $\mathcal{A}$. 

$I[\kappa]$ is a normal ideal on $\kappa$. If $\kappa < \kappa = \kappa$, then there is a maximal (modulo the non-stationary ideal) set in $\mathcal{I}[\kappa]$. This set is called the set of approachable points.
Approachability

Definition
Let $\mu$ be a singular cardinal, $\kappa = \mu^+$, and $\theta$ a sufficiently large, regular cardinal. If $\mathcal{A}$ is a countable expansion of $\langle H(\theta), \in, <_\theta \rangle$ and $\beta < \kappa$ is a limit ordinal, then $\beta$ is approachable with respect to $\mathcal{A}$ if there is an unbounded $\mathcal{A} \subseteq \beta$ of order type $\text{cf}(\beta)$ such that every initial segment of $\mathcal{A}$ is in $\text{Sk}^\mathcal{A}(\beta)$.

Fact
Let $\mu$ be a singular cardinal, and let $\kappa = \mu^+$.

1. $S \in \mathcal{I}[\kappa]$ if and only if there is a sufficiently large, regular $\theta$, a countable expansion $\mathcal{A}$ of $\langle H(\theta), \in, <_\theta \rangle$, and a club $C \subseteq \kappa$ such that, for every $\beta \in S \cap C$, $\beta$ is approachable with respect to $\mathcal{A}$.

2. $\mathcal{I}[\kappa]$ is a normal ideal on $\kappa$. 
Approachability

Definition
Let $\mu$ be a singular cardinal, $\kappa = \mu^+$, and $\theta$ a sufficiently large, regular cardinal. If $A$ is a countable expansion of $\langle H(\theta), \in, <_{\theta} \rangle$ and $\beta < \kappa$ is a limit ordinal, then $\beta$ is approachable with respect to $A$ if there is an unbounded $A \subseteq \beta$ of order type $\text{cf}(\beta)$ such that every initial segment of $A$ is in $\text{Sk}^A(\beta)$.

Fact
Let $\mu$ be a singular cardinal, and let $\kappa = \mu^+$.

1 $S \in I[\kappa]$ if and only if there is a sufficiently large, regular $\theta$, a countable expansion $A$ of $\langle H(\theta), \in, <_{\theta} \rangle$, and a club $C \subseteq \kappa$ such that, for every $\beta \in S \cap C$, $\beta$ is approachable with respect to $A$.

2 $I[\kappa]$ is a normal ideal on $\kappa$.

3 If $\kappa^{<\kappa} = \kappa$, then there is a maximal (modulo the non-stationary ideal) set in $I[\kappa]$. This set is called the set of approachable points.
Approachability and forcing

Theorem (Shelah)

Let $\lambda < \kappa$ be cardinals, with $\kappa$ regular. Suppose that $S \subseteq \kappa \cap \text{cof}(< \lambda)$ is stationary, $S \in l[\kappa]$, and $\mathbb{P}$ is a $\mu$-closed forcing poset. Then $S$ remains stationary in $V^\mathbb{P}$.
Approachability and forcing

Theorem (Shelah)

Let \( \lambda < \kappa \) be cardinals, with \( \kappa \) regular. Suppose that 
\( S \subseteq \kappa \cap \text{cof}(< \lambda) \) is stationary, \( S \in l[\kappa] \), and \( \mathbb{P} \) is a \( \mu \)-closed forcing poset. Then \( S \) remains stationary in \( V^\mathbb{P} \).

Theorem

Let \( \mu \) be a singular cardinal, let \( \kappa = \mu^+ \), and suppose \( \kappa^{<\kappa} = \kappa \). Let \( \bar{a} = \langle a_\alpha \mid \alpha < \kappa \rangle \) be an enumeration of all bounded subsets of \( \kappa \).
Approachability and forcing

Theorem (Shelah)
Let $\lambda < \kappa$ be cardinals, with $\kappa$ regular. Suppose that $S \subseteq \kappa \cap \text{cof}(\prec \lambda)$ is stationary, $S \in I[\kappa]$, and $\mathbb{P}$ is a $\mu$-closed forcing poset. Then $S$ remains stationary in $V^\mathbb{P}$.

Theorem
Let $\mu$ be a singular cardinal, let $\kappa = \mu^+$, and suppose $\kappa^{<\kappa} = \kappa$. Let $\bar{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ be an enumeration of all bounded subsets of $\kappa$. Let $\mathbb{Q}$ be a forcing poset whose conditions are closed, bounded subsets $t$ of $\kappa$ such that, for every $\beta \in t$, $\beta$ is approachable with respect to $\bar{a}$, ordered by end-extension.
Approachability and forcing

Theorem (Shelah)

Let $\lambda < \kappa$ be cardinals, with $\kappa$ regular. Suppose that $S \subseteq \kappa \cap \text{cof}(\lambda)$ is stationary, $S \in l[\kappa]$, and $\mathbb{P}$ is a $\mu$-closed forcing poset. Then $S$ remains stationary in $V^\mathbb{P}$.

Theorem

Let $\mu$ be a singular cardinal, let $\kappa = \mu^+$, and suppose $\kappa^{<\kappa} = \kappa$. Let $\bar{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ be an enumeration of all bounded subsets of $\kappa$. Let $\mathbb{Q}$ be a forcing poset whose conditions are closed, bounded subsets $t$ of $\kappa$ such that, for every $\beta \in t$, $\beta$ is approachable with respect to $\bar{a}$, ordered by end-extension. Then $\mathbb{Q}$ is a cardinal-preserving forcing poset and $V^\mathbb{Q} \models AP_\mu$. 
Supercompact cardinals

Definition

Let $\kappa<\lambda$ be cardinals. $\kappa$ is $\lambda$-supercompact if there is an elementary embedding $j: V \rightarrow M$, where $M$ is a transitive class, such that:

• $\text{crit}(j) = \kappa$.
• $j(\kappa) > \lambda$.
• $\lambda \subseteq M$.

$\kappa$ is supercompact if it is $\lambda$-supercompact for all $\lambda$.

Proposition

$\kappa$ is $\lambda$-supercompact if and only if there is a normal, fine, $\kappa$-complete measure on $P_\kappa(\lambda)$. 
Supercompact cardinals

Definition
Let $\kappa < \lambda$ be cardinals. $\kappa$ is $\lambda$-supercompact if there is an elementary embedding $j : V \rightarrow M$, where $M$ is a transitive class, such that:

- $\text{crit}(j) = \kappa$.
- $j(\kappa) > \lambda$.
- $\lambda^M \subseteq M$. 

Proposition
$\kappa$ is $\lambda$-supercompact if and only if there is a normal, fine, $\kappa$-complete measure on $P^\kappa(\lambda)$. 
Supercompact cardinals

Definition
Let $\kappa < \lambda$ be cardinals. $\kappa$ is $\lambda$-supercompact if there is an elementary embedding $j : V \rightarrow M$, where $M$ is a transitive class, such that:

- $\text{crit}(j) = \kappa$.
- $j(\kappa) > \lambda$.
- $\lambda M \subseteq M$.

$\kappa$ is supercompact if it is $\lambda$-supercompact for all $\lambda$.

Proposition
$\kappa$ is $\lambda$-supercompact if and only if there is a normal, fine, $\kappa$-complete measure on $\mathcal{P}_\kappa(\lambda)$.
Theorem (Solovay)

Suppose $\mu$ is a singular limit of supercompact cardinals. Then every stationary subset of $\mu^+$ reflects.

Proof.
Let $\mu$ be a singular limit of supercompact cardinals, and let $S \subseteq \mu^+$ be stationary. We will show that $S$ reflects.
Theorem (Solovay)

Suppose $\mu$ is a singular limit of supercompact cardinals. Then every stationary subset of $\mu^+$ reflects.

Proof.
Let $\mu$ be a singular limit of supercompact cardinals, and let $S \subseteq \mu^+$ be stationary. We will show that $S$ reflects. By shrinking $S$ if necessary, we may assume that there is $\lambda < \mu$ such that $S \subseteq \mu^+ \cap \operatorname{cof}(\lambda)$. Fix a supercompact cardinal $\kappa$ such that $\lambda < \kappa < \mu$, and let $j : V \to M$ witness that $\kappa$ is $\mu^+$-supercompact.
Theorem (Solovay)

Suppose $\mu$ is a singular limit of supercompact cardinals. Then every stationary subset of $\mu^+$ reflects.

Proof.
Let $\mu$ be a singular limit of supercompact cardinals, and let $S \subseteq \mu^+$ be stationary. We will show that $S$ reflects. By shrinking $S$ if necessary, we may assume that there is $\lambda < \mu$ such that $S \subseteq \mu^+ \cap \text{cof}(\lambda)$. Fix a supercompact cardinal $\kappa$ such that $\lambda < \kappa < \mu$, and let $j : V \to M$ witness that $\kappa$ is $\mu^+$-supercompact.

Let $\gamma = \sup(j^{``}\mu^+)$. Note that, since $j^{``}\mu^+ \in M$ and $M \models ``j(\mu^+)$ is regular $'$, we have $\gamma < j(\mu^+)$. 
Proof ctd.
In \( M \), \( j(S) \) is a stationary subset of \( j(\mu^+) \cap \text{cof}(\lambda) \).
Proof ctd.
In $M$, $j(S)$ is a stationary subset of $j(\mu^+) \cap \text{cof}(\lambda)$.

Claim
$M \models "j(S) \cap \gamma \text{ is stationary in } \gamma."$
Proof ctd.
In $M$, $j(S)$ is a stationary subset of $j(\mu^+) \cap \text{cof}(\lambda)$.

Claim
$M \models "j(S) \cap \gamma \text{ is stationary in } \gamma"$.

Proof.
Suppose not, and let $C \in M$ be a club in $\gamma$ disjoint from $j(S)$. Since $j$ is continuous at ordinals of cofinality $< \kappa$, $D = C \cap j'' \mu^+$ is a $< \kappa$-club in $\gamma$ disjoint from $j(S)$. Thus, $E = j^{-1} "D$ is a $< \kappa$-club in $\mu^+$ disjoint from $S$, contradicting the assumptions that $S$ is a stationary subset of $\mu^+ \cap \text{cof}(\lambda)$ and $\lambda < \kappa$. \qed
Proof ctd.
In $M$, $j(S)$ is a stationary subset of $j(\mu^+) \cap \text{cof}(\lambda)$.

Claim

$M \models \text{"}j(S) \cap \gamma \text{ is stationary in } \gamma\text{"}.$

Proof.
Suppose not, and let $C \in M$ be a club in $\gamma$ disjoint from $j(S)$. Since $j$ is continuous at ordinals of cofinality $< \kappa$, $D = C \cap j''\mu^+$ is a $< \kappa$-club in $\gamma$ disjoint from $j(S)$. Thus, $E = j^{-1}\text{"}D \text{ is a } < \kappa \text{-club in } \mu^+ \text{ disjoint from } S\text{"}$, contradicting the assumptions that $S$ is a stationary subset of $\mu^+ \cap \text{cof}(\lambda)$ and $\lambda < \kappa$. $\square$

Thus, $M \models \text{"}j(S) \text{ reflects at } \gamma\text{"}$, so, by elementarity, $V \models \text{"}S \text{ reflects at some ordinal } < \mu^+\text{"}$. $\square$
Supercompacts, forcing, and reflection

Fact
Suppose $n < \omega$ and every stationary subset of $\aleph_{\omega \cdot n + 1}$ reflects. Then $AP_{\aleph_{\omega \cdot n}}$ holds.
Supercompacts, forcing, and reflection

Fact
Suppose $n < \omega$ and every stationary subset of $\aleph_{\omega \cdot n + 1}$ reflects. Then $AP_{\aleph_{\omega \cdot n}}$ holds.

Theorem (Magidor)
Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of cardinals, with $\kappa_0 = \omega$ and $\kappa_n$ supercompact for all other $n$. Let $\mathbb{P}$ be the full-support iteration of $Coll(\kappa_n, \kappa_{n+1})$ for $n < \omega$. Then, in $V^\mathbb{P}$, every stationary subset of $\aleph_{\omega + 1}$ reflects.
Supercompacts, forcing, and reflection

Fact
Suppose $n < \omega$ and every stationary subset of $\aleph_{\omega \cdot n + 1}$ reflects. Then $AP_{\aleph_{\omega \cdot n}}$ holds.

Theorem (Magidor)
Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of cardinals, with $\kappa_0 = \omega$ and $\kappa_n$ supercompact for all other $n$. Let $P$ be the full-support iteration of $\text{Coll}(\kappa_n, \kappa_{n+1})$ for $n < \omega$. Then, in $V^P$, every stationary subset of $\aleph_{\omega + 1}$ reflects.

Theorem (Chayut)
Let $\langle \kappa_n \mid n < \omega \rangle$ be as above, and let $\mu = \sup(\{\kappa_n \mid n < \omega\})$. Let $\langle \lambda_n \mid n < \omega \rangle$ be a sequence of regular cardinals such that, for all $n < \omega$, $\kappa_n \leq \lambda_n < \kappa_{n+1}$. Let $P$ be the full-support iteration of $\text{Coll}(\lambda_n, \kappa_{n+1})$ and, in $V^P$, let $Q$ be the poset to force $AP_\mu$. Then, in $V^{P \ast Q}$, every stationary subset of $\mu^+$ reflects.
Section 3

Bounded stationary reflection
Bounded stationary reflection

Definition
Let $\mu$ be a singular cardinal, and let $\kappa = \mu^+$. Bounded stationary reflection holds at $\kappa$ if every stationary subset of $\kappa$ reflects, but there is a stationary $S \subseteq \kappa$ and a $\lambda < \mu$ such that $S$ does not reflect at any ordinals in $\kappa \cap \text{cof}(\geq \lambda)$.

Fact
Bounded stationary reflection cannot hold at $\aleph_{\omega+1}$.
Bounded stationary reflection

Definition
Let $\mu$ be a singular cardinal, and let $\kappa = \mu^+$. *Bounded stationary reflection* holds at $\kappa$ if every stationary subset of $\kappa$ reflects, but there is a stationary $S \subseteq \kappa$ and a $\lambda < \mu$ such that $S$ does not reflect at any ordinals in $\kappa \cap \text{cof}(\geq \lambda)$.

Fact
*Bounded stationary reflection cannot hold at $\aleph_{\omega + 1}$*. 
Bounded stationary reflection

Theorem (Cummings, L-H)
Assume there are $\omega \cdot 2$-many supercompact cardinals. Then there is a forcing extension in which bounded stationary reflection holds at $\aleph_{\omega \cdot 2 + 1}$.
Bounded stationary reflection

Theorem (Cummings, L-H)

Assume there are $\omega \cdot 2$-many supercompact cardinals. Then there is a forcing extension in which bounded stationary reflection holds at $\aleph_{\omega \cdot 2 + 1}$.

Proof sketch.
Let $\langle \kappa_i \mid i \leq \omega \cdot 2 + 1 \rangle$ be an increasing, continuous sequence of cardinals such that:

- $\kappa_0 = \omega$.
- If $0 \leq i < \omega$ or $\omega < i < \omega \cdot 2$, then $\kappa_{i+1}$ is supercompact.
- $\kappa_{\omega+1} = \kappa_\omega^+$ and $\kappa_{\omega \cdot 2 + 1} = \kappa_{\omega \cdot 2}^+$. 
Bounded stationary reflection

Theorem (Cummings, L-H)
Assume there are $\omega \cdot 2$-many supercompact cardinals. Then there is a forcing extension in which bounded stationary reflection holds at $\aleph_{\omega \cdot 2 + 1}$.

Proof sketch.
Let $\langle \kappa_i \mid i \leq \omega \cdot 2 + 1 \rangle$ be an increasing, continuous sequence of cardinals such that:

- $\kappa_0 = \omega$.
- If $0 \leq i < \omega$ or $\omega < i < \omega \cdot 2$, then $\kappa_{i+1}$ is supercompact.
- $\kappa_{\omega+1} = \kappa_{\omega}^+$ and $\kappa_{\omega \cdot 2 + 1} = \kappa_{\omega \cdot 2}^+$.

Let $\mathbb{P} = \mathbb{P}_0 \ast \dot{\mathbb{P}}_1$, where $\mathbb{P}_0$ is the full-support iteration of $\text{Coll}(\kappa_i, \kappa_{i+1})$ for $i < \omega$ and, in $V^{\mathbb{P}_0}$, $\mathbb{P}_1$ is the full-support iteration of $\text{Coll}(\kappa_i, \kappa_{i+1})$ for $\omega + 1 \leq i < \omega \cdot 2$. 
In $V^P$, we have $\kappa_i = \aleph_i$ for all $i \leq \omega \cdot 2 + 1$. Let $\kappa = \kappa_{\omega \cdot 2 + 1}$. We also have $\kappa <^\kappa = \kappa$. Fix an enumeration $\bar{a}$ of all bounded subsets of $\kappa$ in order type $\kappa$, and let $\mathbb{Q}$ be the forcing to shoot a club through the set of ordinals that are approachable with respect to $\bar{a}$.\[\text{In } V^P, \text{ we have } \kappa_i = \aleph_i \text{ for all } i \leq \omega \cdot 2 + 1. \text{ Let } \kappa = \kappa_{\omega \cdot 2 + 1}. \text{ We also have } \kappa <^\kappa = \kappa. \text{ Fix an enumeration } \bar{a} \text{ of all bounded subsets of } \kappa \text{ in order type } \kappa, \text{ and let } \mathbb{Q} \text{ be the forcing to shoot a club through the set of ordinals that are approachable with respect to } \bar{a}.\]
In $V^P$, we have $\kappa_i = \aleph_i$ for all $i \leq \omega \cdot 2 + 1$. Let $\kappa = \kappa_{\omega \cdot 2 + 1}$. We also have $\kappa^{<\kappa} = \kappa$. Fix an enumeration $\vec{a}$ of all bounded subsets of $\kappa$ in order type $\kappa$, and let $\mathcal{Q}$ be the forcing to shoot a club through the set of ordinals that are approachable with respect to $\vec{a}$.

In $V^P \ast \dot{\mathcal{Q}}$, every stationary subset of $\kappa$ reflects. However, bounded stationary reflection necessarily fails at $\kappa$. Thus, let $\mathcal{S}$ be the forcing poset whose conditions are bounded subsets $s \subseteq \kappa \cap \text{cof}(\omega)$ such that, for all $\alpha \in \kappa \cap \text{cof}(\geq \kappa_{\omega + 1})$, $\mathcal{S} \cap \alpha$ is not stationary in $\alpha$. $\mathcal{S}$ is ordered by end-extension.
In $V^P$, we have $\kappa_i = \aleph_i$ for all $i \leq \omega \cdot 2 + 1$. Let $\kappa = \kappa_{\omega \cdot 2 + 1}$. We also have $\kappa^{<\kappa} = \kappa$. Fix an enumeration $\bar{a}$ of all bounded subsets of $\kappa$ in order type $\kappa$, and let $Q$ be the forcing to shoot a club through the set of ordinals that are approachable with respect to $\bar{a}$.

In $V^P \dot{\times} Q$, every stationary subset of $\kappa$ reflects. However, bounded stationary reflection necessarily fails at $\kappa$. Thus, let $S$ be the forcing poset whose conditions are bounded subsets $s \subseteq \kappa \cap \text{cof}(\omega)$ such that, for all $\alpha \in \kappa \cap \text{cof}(\geq \kappa_{\omega + 1})$, $S \cap \alpha$ is not stationary in $\alpha$. $S$ is ordered by end-extension. $S$ is a cardinal-preserving forcing poset that adds a stationary subset of $\kappa$ that does not reflect at any ordinals in $\kappa \cap \text{cof}(\geq \kappa_{\omega + 1})$. With a bit of work, one can show that, in $V^P \dot{\times} Q \dot{\times} S$, it is still the case that every stationary subset of $\kappa$ reflects. Thus, bounded stationary reflection holds at $\kappa = \aleph_{\omega \cdot 2 + 1}$.
Theorem (Cummings, L-H)

Suppose there is a proper class of supercompact cardinals. Then there is a class forcing extension in which, for every singular cardinal $\mu > \aleph_\omega$ that is not a cardinal fixed point, bounded stationary reflection holds at $\mu^+$. 
Thank you!