On commuting matrices in max algebra and in classical nonnegative algebra

Hans Schneider
joint with Ricardo Katz and Sergei Sergeev
with input from Peter Butkovic

Applied Linear Algebra
Novi Sad
24 May 2010
Three classical theorems

**Theorem**

*If* $AB = BA$ *then the eigenvalues* $\alpha^j, \beta^j$ *of* $A, B$ *can be ordered so that for any polynomial* $p(x, y)$ *the eigenvalues of* $p(A, B)$ *are* $p(\alpha^j, \beta^j)$, $j = 1, \ldots, n$.

Frobenius 1878
Three classical theorems

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If $AB = BA$ then $A$ and $B$ have a common eigenvector.
We have stated results in

- **CM**: complex linear algebra
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Our aim is to provide analogs of these results in

- **NN**: (classical) nonnegative linear algebra
- **MX**: max (nonnegative) linear algebra
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Stephane Gaubert 1997:
The spectral theory in **MX** "is extremely similar to the well-known Perron-Frobenius theory" in **NN"
We have stated results in

- **CM**: complex linear algebra

Our aim is to provide analogs of these results in

- **NN**: (classical) nonnegative linear algebra
- **MX**: max (nonnegative) linear algebra

Stephane Gaubert 1997:
The spectral theory in **MX** "is extremely similar to the well-known Perron-Frobenius theory" in **NN** with some important differences.
Definition

Eigenvalue $\alpha$ of $A$ is a distinguished eigenvalue if there is an associated nonnegative eigenvector.

*eigenvalue = distinguished eigenvalue
*eigenvector = nonnegative eigenvector
A reducible:

\[ P^{-1}AP = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \]

\( A_{11}, \ A_{22} \) square
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**Theorem**

Let \( A \geq 0 \) be irreducible. Then \( A \) has a unique *eigenvalue \( \lambda(A) \) with an (ess) unique associated *eigenvector, which is positive.

\( \lambda(A) \) is the Perron root
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**Theorem**

Let \( A \geq 0 \). Then \( A \) has *eigenvalue(s) (and *eigenvectors)

Denote the largest *eigenvalue of \( A \) by \( \lambda(A) \)
\[ a, b \geq 0 \]
\[ a \oplus b = \max(a, b) \]
\[ a \otimes b = ab \]
\( a, b \geq 0 \)

\( a \oplus b = \max(a, b) \)

\( a \otimes b = ab \)

\( A, B \in \mathbb{R}_{+}^{n \times n} \)

\( C = A \oplus B : \quad c_{ij} = a_{ij} \oplus b_{ij} \)

\( C = A \otimes B : \quad c_{ij} = \bigoplus_{k} a_{ik} b_{kj} \)
Max algebra: $+$ is MAX!

$a, b \geq 0$

$a + b = \max(a, b)$

$ab = ab$
MAX ALGEBRA: + is MAX!

\[ a, b \geq 0 \]

\[ a + b = \max(a, b) \]

\[ ab = ab \]

\[ C = A + B : \quad c_{ij} = a_{ij} + b_{ij} \]

\[ C = AB : \quad c_{ij} = \bigoplus_{k} a_{ik} b_{kj} \]
Definition

α is an eigenvalue of A:
∃x ⪈ 0, Ax = αx
x is an eigenvector corr. α
Definition

\( \alpha \) is an eigenvalue of \( A \):
\[ \exists x \geq 0, Ax = \alpha x \]
x is an eigenvector corr. \( \alpha \)

Theorem

Let \( A \geq 0 \) be irreducible. Then \( A \) has a unique eigenvalue \( \lambda(A) \) with associated eigenvectorS, which are positive.

\( \lambda(A) \) is the Perron root

max cycle geom mean
Warning

\[ A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{3}{4} & 1 \end{pmatrix} \]

\[ (A^2)_{11} = \max\{1, \frac{3}{8}\} = 1 \]

\[ (A^2)_{21} = \max\{\frac{3}{4}, \frac{3}{4}\} = \frac{3}{4} \]

\[ A^2 = A \]
A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{3}{4} & 1 \end{pmatrix}

(A^2)_{11} = \max\{1, \frac{3}{8}\} = 1

(A^2)_{21} = \max\{\frac{3}{4}, \frac{3}{4}\} = \frac{3}{4}

A^2 = A

Both columns are eigenvectors
Warning

$$A = \begin{pmatrix} 1 & 1 \\ 3/4 & 1/2 \end{pmatrix}$$

$$(A^2)_{11} = \max\{1, 3/8\} = 1$$

$$(A^2)_{21} = \max\{3/4, 3/4\} = 3/4$$

$$A^2 = A$$

Both columns are eigenvectors

**Theorem**

*Let $A \geq 0$. Then $A$ has eigenvalue(s) (and eigenvectors)*
**CM**: \[
\begin{pmatrix}
4 & 0 & 0 \\
3 & 1 & 0 \\
1 & 1 & 2
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & -1 & 1
\end{pmatrix}
\]
<table>
<thead>
<tr>
<th>CM</th>
<th>NN</th>
<th>MX</th>
</tr>
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</table>
| \[
\begin{pmatrix}
4 & 0 & 0 \\
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1 & 0 \\
1 & 0 \\
1 & 1
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 0 \\
1 & 0 \\
1 & 1
\end{pmatrix}
\] |
<table>
<thead>
<tr>
<th></th>
<th>Matrix</th>
<th>Eigenvectors</th>
</tr>
</thead>
</table>
| **CM:** | \[
\begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}
\] | \[
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\] |
| **NN:** | \[
\begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}
\] | \[
\begin{pmatrix} 1 & . & 0 \\ 1 & . & 0 \\ 1 & . & 1 \end{pmatrix}
\] |
| **MX:** | \[
\begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}
\] | \[
\begin{pmatrix} 4 & . & 0 \\ 3 & . & 0 \\ 1 & . & 1 \end{pmatrix}
\] |
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| \[
\begin{pmatrix}
4 & 0 & 0 \\
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1 & 1 & 2
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\] | \[
\begin{pmatrix}
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\end{pmatrix}
\] |

<table>
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</tr>
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</table>
| \[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\] | \[
\begin{pmatrix}
1.0 \\
1.0 \\
1.1
\end{pmatrix}
\] | \[
\begin{pmatrix}
4.0 \\
3.0 \\
1.1
\end{pmatrix}
\] |

**NN, MX**: evalues 4, 2
Theorem

If $AB = BA$ then $A$ and $B$ have a common eigenvector.
Theorem

If $AB = BA$ then $A$ and $B$ have a common eigenvector.

**CM:** Complex matrices – $X$ basis of espace for evalue $\alpha$

**Proof.**

\[
AX = \alpha X
\]

\[
A(BX) = B(AX) = \alpha BX
\]

\[
BX = XC
\]

\[
Cz = \beta z, \quad z \neq 0
\]

\[
B(Xz) = X(Cz) = \beta Xz, \quad Xz \neq 0
\]

\[
A(Xz) = (AX)z = \alpha Xz
\]
One theorem, three incarnations, one proof

**Theorem**

*If* $AB = BA$ *then* $A$ *and* $B$ *have a common eigenvector.*

**NN:** classic nonneg – $X$ extremals of convex econe for evalue $\alpha$

**Proof.**

$$AX = \alpha X$$

$$A(BX) = B(AX) = \alpha BX$$

$$BX = XC$$

$$Cz = \beta z \quad z \neq 0$$

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$$A(Xz) = (AX)z = \alpha Xz$$
Theorem

If $AB = BA$ then $A$ and $B$ have a common eigenvector.

**MX:** max nonneg – $X$ extremals of max econe for evale $\alpha$

**Proof.**

\[
AX = \alpha X \\
A(BX) = B(AX) = \alpha BX \\
BX = XC \\
Cz = \beta z \quad z \neq 0 \\
B(Xz) = X(Cz) = \beta Xz, \quad Xz \neq 0 \\
A(Xz) = (AX)z = \alpha Xz
\]
Theorem

Let $A_1, \ldots, A_r$ be pairwise commuting matrices. Then for each eigenvalue $\alpha^j$ of $A_j$ there exists an eigenvector $x$ which is an eigenvector of all the $A_j$. 
Theorem

If $AX = XB$ and

**CM:** the cols of $X$ are lin indep 
**NN & MX:** no col of $X$ is 0

then every eval of $B$ is an eval of $A$.

Proof.

$Bz = \beta z$, $z \neq 0$

$AXz = XBz = \beta Xz$, $Xz \neq 0$
Theorem

**CM:** Suppose that $A_1, \ldots, A_r \in \mathbb{R}^{n \times n}_+$ pairwise commute. For $i = 1, \ldots, r$, let the eigenvalues of $A_i$ be $\alpha^i_j$ for $j = 1, \ldots, n$. Let $p(x_1, \ldots, x_r)$ be a polynomial. Then, the eigenvalues $\alpha^i_j$ can be ordered so that the eigenvalues of $p(A_1, \ldots, A_r)$ are $p(\alpha^1_j, \ldots, \alpha^r_j)$ for $j = 1, \ldots, n$. Frobenius 1896, Schur 1902
Theorem

**MX: & NN** Let $A_1, \ldots, A_r \in \mathbb{R}^{n \times n}_+$ commute in pairs and let $p(x_1, \ldots, x_r)$ be a polynomial such that $p(A_1, \ldots, A_r) \geq 0$

Then,

(i) For each $i \in \{1, \ldots, r\}$ and evale $\alpha_i$ of $A_i$ there exist evales $\alpha_j$ of $A_j$ for all $j \neq i$ such that $p(\alpha_1, \ldots, \alpha_r)$ is an evale of $p(A_1, \ldots, A_r)$;

(ii) For each evale $\lambda$ of $p(A_1, \ldots, A_r)$ there exist evales $\alpha_i$ of $A_i$ for all $i = 1, \ldots, r$ such that $\lambda = p(\alpha_1, \ldots, \alpha_r)$. 
Frobenius form 1912

\[ A \leftarrow P^{-1} A P = \begin{pmatrix}
    A_{11} & 0 & \cdots & 0 & 0 \\
    A_{21} & A_{22} & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    A_{(k-1)1} & A_{(k-1)2} & \cdots & A_{(k-1)(k-1)} & 0 \\
    A_{k1} & A_{k2} & \cdots & A_{k(k-1)} & A_{kk}
\end{pmatrix} \]

\( A_{ii} \) irreducible
Reduced graph \( \mathcal{R}(A) \)

\[ V = \{1, \ldots, k\} \]

\[ i \rightarrow j \in E : A_{ij} \neq 0 \]

Path from \( i \) to \( j \)

\[ i_0 \rightarrow i_1 \rightarrow \cdots \rightarrow i_{p-1} \rightarrow i_m \]

Transitive closure \( \mathcal{R}^*(A) \)

\[ i \rightarrow^* j : \text{exists path from } i \text{ to } j \]

Skeleton \( S = \mathcal{R}^*_s(A) \)

\[ (i,j) \in S : i \rightarrow^* k \rightarrow^* j \text{ implies } k = i \text{ or } k = j \]
Example

\[
\begin{pmatrix}
\spadesuit & 0 & 0 & 0 \\
\clubsuit & \spadesuit & 0 & 0 \\
0 & \spadesuit & \spadesuit & 0 \\
\clubsuit & \clubsuit & 0 & \spadesuit \\
\end{pmatrix}
\]

\spadesuit \quad \text{irred block}

\clubsuit \quad \text{nonzero block}
Example

\[
\begin{pmatrix}
\spadesuit & 0 & 0 & 0 \\
\clubsuit & \spadesuit & 0 & 0 \\
0 & \spadesuit & \spadesuit & 0 \\
\spadesuit & \spadesuit & \spadesuit & 0
\end{pmatrix}
\]

\spadesuit \quad \text{irred block}

\clubsuit \quad \text{nonzero block}

(1) \quad \leftrightarrow \quad (2) \quad \leftrightarrow \quad (3)

(4)
Example

\[
\begin{pmatrix}
\spadesuit & 0 & 0 & 0 \\
\clubsuit & \spadesuit & 0 & 0 \\
0 & \spadesuit & \spadesuit & 0 \\
\clubsuit & \clubsuit & 0 & \spadesuit \\
\end{pmatrix}
\]

\spadesuit irred block
\clubsuit nonzero block

(1) $\leftarrow$ (2) $\leftarrow$ (3)

(4)

\[
\begin{pmatrix}
\spadesuit & 0 & 0 & 0 \\
\clubsuit & \spadesuit & 0 & 0 \\
\heartsuit & \spadesuit & \spadesuit & 0 \\
\heartsuit & \clubsuit & \spadesuit & 0 \\
\end{pmatrix}
\]

\heartsuit in trans closure of skeleton

Katz, Schneider, Sergeev
Commuting Matrices
Identify:
node of $R(A)$ - irred block of $A$

CLASS of $A$

$(1) \leftarrow (2) \leftarrow (3) \uparrow (4)$
Identify:

node of $\mathcal{R}(A)$ - irreducible block of $A$
Identify: node of $R(A)$ - irred block of $A$
CLASS of $A$

$\begin{pmatrix}
\heartsuit & 0 & 0 & 0 \\
\spadesuit & \spadesuit & 0 & 0 \\
\spadesuit & \spadesuit & \spadesuit & 0 \\
\heartsuit & \clubsuit & 0 & \spadesuit \\
\end{pmatrix}$

(1) ←− (2) ←− (3)
(4)

(♠)
(♣) ←− (♣) ←− (♣)
(♣)
Definition

A class $A_{ii}$ of $A$ is called *spectral* if $\lambda(A_{ii})$ is an eigenvalue of $A$ and there is a evector $x$ such that $x_i \neq 0$ if and only if $i \to^* j$ in $R^*(A)$
**Definition**

A class $A_{jj}$ of $A$ is called *spectral* if $\lambda(A_{jj})$ is an eigenvalue of $A$ and there is a evector $x$ such that $x_i \neq 0$ if and only if $i \to^* j$ in $R^*(A)$.

**Theorem**

Assume that $A$ is in Frobenius form with Perron root $\alpha_j$ of $A_{jj}$. Then

*(Frobenius 1912, Victory 1985)*

*MX:* $A_{jj}$ is spectral if and only if $i \to^* j$ in $R^*(A)$ implies that $\alpha_i \leq \alpha_j$.

*Gaubert 1992, Butkovic (book) 2010*
**Definition**

A class $A_{ji}$ of $A$ is called *spectral* if $\lambda(A_{ji})$ is an eigenvalue of $A$ and there is a evector $x$ such that $x_i \neq 0$ if and only if $i \rightarrow^* j$ in $R^*(A)$.

**Theorem**

Assume that $A$ is in Frobenius form with Perron root $\alpha_j$ of $A_{jj}$. Then

**NN:** $A_{jj}$ is spectral if and only if $i \rightarrow^* j$ in $R^*(A)$ implies that $\alpha_i < \alpha_j$.

(Frobenius 1912, Victory 1985)
spectral classes

**Definition**

A class $A_{ii}$ of $A$ is called *spectral* if $\lambda(A_{ii})$ is an eigenvalue of $A$ and there is a evector $x$ such that $x_i \neq 0$ if and only if $i \rightarrow j$ in $\mathcal{R}^*(A)$.

**Theorem**

Assume that $A$ is in Frobenius form with Perron root $\alpha_j$ of $A_{jj}$. Then

**NN:** $A_{jj}$ is spectral if and only if $i \rightarrow j$ in $\mathcal{R}^*(A)$ implies that $\alpha_i < \alpha_j$.

(Frobenius 1912, Victory 1985)

**MX:** $A_{jj}$ is spectral if and only if $i \rightarrow j$ in $\mathcal{R}^*(A)$ implies that $\alpha_i \leq \alpha_j$.

Gaubert 1992, Butkovic (book) 2010
Theorem (NN:, MX:)

Assume that $A$ is in Frobenius form with Perron roots $\alpha_k$ of $A_{kk}$ pairwise distinct. Let $A_{jj}$ be a spectral class of $A$. Then there exists an eigenvector $x$ such that $x_i \neq 0$ if and only if $i \rightarrow j$.
Theorem (\textbf{NN:}, \textbf{MX:})

Assume that $A$ is in Frobenius form with Perron roots $\alpha_k$ of $A_{kk}$ pairwise distinct. Let $A_{jj}$ be a spectral class of $A$. Then there exists an eigenvector $x$ such that $x_i \neq 0$ if and only if $i \rightarrow^* j$. 

\begin{align*}
\begin{bmatrix}
10 \\
\ast \\
0 \\
0 \\
5 \\
0 \\
0 \\
2 \\
3 \\
3 \\
\ast
\end{bmatrix}
\begin{bmatrix}
2 \\
0 \\
1 \\
0 \\
1 \\
0 \\
\ast \\
\ast
\end{bmatrix}
\end{align*}
Theorem (NN:, MX:)

Assume that $A$ is in Frobenius form with Perron roots $\alpha_k$ of $A_{kk}$ pairwise distinct. Let $A_{jj}$ be a spectral class of $A$. Then there exists an eigenvector $x$ such that $x_i \neq 0$ if and only if $i \to j$.

\[
\begin{pmatrix}
10^* & 0 & 0 \\
5 & 0 & 0 \\
2 & 3 & 3^*
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
1 & 0 \\
1 & 1
\end{pmatrix}
\]

Skeleton *Spectral

\((^*1) \leftarrow (2) \leftarrow (^*3)\)
Theorem

NN: & (MX: + is MAX !)

Suppose that $A_1, \ldots, A_r \in \mathbb{R}^{n \times n}_+$ pairwise commute and that distinct classes of $A_i$, for $i = 1, \ldots, r$, have distinct Perron roots. Then,
Theorem

NN : & (MX: + is MAX !)

Suppose that $A_1, \ldots, A_r \in \mathbb{R}_+^{n\times n}$ pairwise commute and that distinct classes of $A_i$, for $i = 1, \ldots, r$, have distinct Perron roots. Then,

1. The classes of $A_1, \ldots, A_r$ and $A_1 + \ldots + A_r$ coincide.
Theorem

Suppose that $A_1, \ldots, A_r \in \mathbb{R}^{n \times n}_+$ pairwise commute and that distinct classes of $A_i$, for $i = 1, \ldots, r$, have distinct Perron roots. Then,

1. The classes of $A_1, \ldots, A_r$ and $A_1 + \ldots + A_r$ coincide.
2. The transitive closures of the reduced graphs of all matrices $A_1, \ldots, A_r$ and $A_1 + \ldots + A_r$ coincide.
Theorem

3. The spectral classes of the matrices $A_1, \ldots, A_r$ coincide
   **MX:** and also coincide with the spectral classes of
   $A_1, \ldots, A_r$ and $A_1 + \ldots + A_r$.
   In particular, $A_1, \ldots, A_r$ have the same number of
   distinct eigenvalues, which we denote by $m$. 
Theorem

4. For $i = 1, \ldots, r$, let the (distinct) eigenvalues of $A_{ii}$ be $\alpha^i_j$ for $j = 1, \ldots, m$.

**MX:** Let $p(x_1, \ldots, x_r)$ be a non-constant max-polynomial.

**NN:** Let $p(x_1, \ldots, x_r)$ be a non-constant polynomial such that $p(A_1, \ldots, A_r) \geq 0$.

Then, the eigenvalues $\alpha^i_j$ can be ordered so that the eigenvalues of $p(A_1, \ldots, A_r)$ are precisely $p(\alpha^1_j, \ldots, \alpha^r_j)$ for $j = 1, \ldots, m$. 
Assume in Frobenius form, \( B_{11} \) and \( B_{22} \) no common evalue.

Compare \((AB)_{21}\) and \((BA)_{21}\)

Assume \( B_{21} = 0 \)

\[
(AB)_{21} = A_{21}B_{11}
\]

\[
(BA)_{21} = B_{22}A_{21}
\]

\[
A_{21} = 0
\]
\[
A = \begin{pmatrix}
10 & 0 & 0 \\
5 & 0 & 0 \\
2 & 3 & 3
\end{pmatrix} \quad B = \begin{pmatrix}
3 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 2
\end{pmatrix}
\]
\[
A = \begin{pmatrix} 10 & 0 & 0 \\ 5 & 0 & 0 \\ 2 & 3 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}
\]

Skeleton \quad *Spectral

\((*1) \leftrightarrow (2) \leftrightarrow (*3)\)
\[ A = \begin{pmatrix} 10 & 0 & 0 \\ 5 & 0 & 0 \\ 2 & 3 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \]

Skeleton \quad \begin{array}{c} \text{*Spectral} \\ (*1) \leftrightarrow (2) \leftrightarrow (*3) \end{array}

\[ AB = BA = \begin{pmatrix} 30 & 0 & 0 \\ 15 & 0 & 0 \\ 9 & 6 & 6 \end{pmatrix} \]
Example

\[
A = \begin{pmatrix}
10 & 0 & 0 \\
5 & 0 & 0 \\
2 & 3 & 3 \\
\end{pmatrix}
\quad B = \begin{pmatrix}
3 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 2 \\
\end{pmatrix}
\]

Skeleton, *Spectral

\[
(1^*) \leftrightarrow (2) \leftrightarrow (3^*)
\]
\[
A = \begin{pmatrix}
10 & 0 & 0 \\
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\end{pmatrix} \quad B = \begin{pmatrix}
3 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 2 \\
\end{pmatrix}
\]

Skeleton, *Spectral

\[(1^*) \leftarrow (2) \leftarrow (3^*)\]

*Eigenvectors of \( A \), \( B \) and \( AB \).
If you’re interested, see

R. Katz, H. Schneider, S. Sergeev
On commuting matrices in max algebra
and in classical nonnegative algebra

http://www.math.wisc.edu/~hans/
My Papers
Paper 160
THANKS
THANKS
FOR THE HONOR YOU HAVE DONE ME
THANKS
FOR THE HONOR YOU HAVE DONE ME
and even more
THANKS
FOR THE HONOR YOU HAVE DONE ME
and even more
FOR LISTENING TO ME
THANKS
FOR THE HONOR YOU HAVE DONE ME
and even more
FOR LISTENING TO ME
If you listened