Tukey Degrees of Ultrafilters

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joint work with

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\textbf{Def.} $\mathcal{U} \leq_T \mathcal{V}$ iff there is a \textit{Tukey} map $g : \mathcal{U} \rightarrow \mathcal{V}$ taking unbounded subsets of $\mathcal{U}$ to unbounded subsets of $\mathcal{V}$.

Equivalently, $\mathcal{U} \leq_T \mathcal{V}$ iff there is a \textit{cofinal} map $f : \mathcal{V} \rightarrow \mathcal{U}$ taking cofinal subsets of $\mathcal{U}$ to cofinal subsets of $\mathcal{V}$.

$\mathcal{U} \equiv_T \mathcal{V}$ iff $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$.

\textbf{Fact.} $\equiv_T$ is an equivalence relation. $\leq_T$ is a partial ordering on the equivalence classes.
Motivations

1. A special class of directed systems of size $\mathfrak{c}$.

2. $\nu \geq_{RK} \mathcal{U}$ implies $\nu \geq_{T} \mathcal{U}$.

What is the structure of Tukey degrees of ultrafilters on $\omega$?
[Isbell 65] There is an ultrafilter \( U_{top} \equiv_T [\mathfrak{c}]^{<\omega} \).

Note: \( V \equiv_T [\mathfrak{c}]^{<\omega} \) iff \( \neg (\forall S \in \mathcal{V}^c \exists T \in [S]^\omega (\bigcap T \in V)) \).

**Question.** [Isbell 65] Is there always (in ZFC) an ultrafilter \( U \) such that \( U <_T U_{top} \)?
Note: $\mathcal{V} \equiv_T [c]<^\omega$ iff $(\forall S \in [\mathcal{V}]^c \exists T \in [S]^\omega (\bigcap T \in \mathcal{V}))$.

**Def.** [Solecki/Todorcevic 04] An ultrafilter $\mathcal{V}$ is *basic* if each convergent sequence has a bounded subsequence.

**Fact.** Each basic ultrafilter does not have top Tukey degree.
Note: \( \mathcal{V} \equiv_T [c]^<\omega \) iff \( \neg (\forall S \in [\mathcal{V}]^c \exists T \in [S]^\omega \ (\cap T \in \mathcal{V})) \).

**Def.** An ultrafilter \( \mathcal{V} \) is *basic* if each convergent sequence has a bounded subsequence.

**Fact.** A basic ultrafilter is does not have top Tukey degree.

**Thm.** An ultrafilter is basic iff it is a p-point.
Are there Tukey non-top ultrafilters which are not p-points?
**Def.** \( U \) is *basically generated* if there is a filter base \( \mathcal{B} \subseteq U \) 
(\( \forall X \in U \ \exists Y \in \mathcal{B} \ Y \subseteq X \)) such that whenever \( A, A_n \in \mathcal{B} \) and \( A_n \to A \), then there is a subsequence such that \( \bigcap_{k<\omega} A_{n_k} \in U \).

**Fact.** A basically generated ultrafilter is not Tukey top.

**Thm.** If \( U, U_n \) are p-points, then \( \lim_{n \to U} U_n \) is basically generated (but not a p-point).

**Fact.** Glazer selective ultrafilters on FIN are basically generated.
We now focus on the structure of Tukey degrees of p-points and ultrafilters below them.
**Theorem.** If $\mathcal{U}$ is a p-point and $\mathcal{U} \gtrsim_T \mathcal{V}$, then there is a continuous monotone map $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $f \upharpoonright \mathcal{U} : \mathcal{U} \to \mathcal{V}$ is a cofinal map.

Note: $f$ is definable from its values on the Fréchet filter.
**Thm.** Every family of p-points of cardinality \( > \aleph^+ \) contains a subfamily of equal size of pairwise Tukey incomparable p-points.

**Thm.** Every \( \leq_T \) chain of p-points has cardinality \( \leq \aleph^+ \).

**Thm.** If \( U \geq_T V \) and \( U \) is selective, then \( V \) is basically generated.
Antichains

**Thm.** 1. If $\text{cov}(\mathcal{M}) = c$, and $2^{<\kappa} = c$, then there are $2^\kappa$ pairwise incomparable selective ultrafilters.

2. If $\mathfrak{d} = \mathfrak{u} = c$ and $2^{<\kappa} = c$, then there are $2^\kappa$ pairwise incomparable p-points.
Chains

[Kunen 78] If $\mathcal{U}$ is $\kappa$-OK and $\kappa > \text{cof}(\mathcal{U})$, then $\mathcal{U}$ is a p-point.

[Milovich 08] $\mathcal{U}$ is a p-point iff it is $\mathfrak{c}$-OK and not Tukey top.

Fact. If $\mathcal{U}$ is $\kappa$-OK but not a p-point, then $\mathcal{U} \geq_T [\kappa]^\omega$. Hence, if $\mathcal{U}$ is $\kappa$-OK but not a p-point, then $\text{cof}(\mathcal{U}) = \kappa$ iff $\mathcal{U} \equiv_T [\kappa]^\omega$.

If there are $\kappa$-OK non p-points with cofinality $\kappa$ for each uncountable $\kappa < \mathfrak{c}$, then there is a strictly increasing chain of ultrafilters of length $\alpha$, where $\alpha$ is such that $\aleph_{\alpha} = \mathfrak{c}$. 
Thm. (also independently by Dilip Raghavan) CH implies for each p-point $\mathcal{U}$ there is a p-point $\mathcal{V}$ such that $\mathcal{V} >_T \mathcal{U}$.

Cor. (CH) There is a Tukey strictly increasing chain of p-points of length $\omega_1$.

Question. Is it true that there is a p-point Tukey above ANY Tukey strictly increasing chain of p-points?
Incomparable Predecessors

**Thm.** (MA) There is a p-point with 2 Tukey incomparable predecessors, each of which is also a p-point.

**Thm.** (CH) There is a Glazer selective ultrafilter $\mathcal{U}$ on FIN such that $\mathcal{U}_{\text{min}}$ and $\mathcal{U}_{\text{max}}$ are Tukey incomparable selective ultrafilters.
Comparing with $\omega^\omega$.

**Fact.** If $\mathcal{U}$ is rapid, then $\mathcal{U} \geq T \omega^\omega$.

**Fact.** For each ultrafilter $\mathcal{U}$, $\mathcal{U} \cdot \mathcal{U} \geq T \omega^\omega$.

**Fact.** If $\mathcal{U}$ is a p-point, then $\mathcal{U}^\omega \equiv T \mathcal{U} \times \omega^\omega$.

**Thm.** If $\mathcal{U}$ is a p-point, then $\mathcal{U} \cdot \mathcal{U} \leq T \mathcal{U}^\omega$.

**Thm.** The following are equivalent for a p-point $\mathcal{U}$

1. $\mathcal{U} \geq_T \omega^\omega$;
2. $\mathcal{U} \equiv_T \mathcal{U} \cdot \mathcal{U}$;
3. $\mathcal{U} \equiv_T \mathcal{U}^\omega$. 
Cor. If \( U \) is a rapid ultrafilter then \( U \cdot U \equiv_T U \).

Cor. If \( U \) is a p-point and \( U \geq_T \omega^\omega \), then \( U \cdot U \equiv_T U \).

Cor. If \( U \) is a p-point of cofinality \(< \delta\), then \( U \nleq_T \omega^\omega \) and therefore \( U \cdot U >_T U \).

Thm. Assuming \( p = c \), there is a p-point \( U \) such that \( U \nleq_T \omega^\omega \) and therefore \( U <_T U \cdot U <_T U_{top} \).
Some Open Problems

1. Is it true in ZFC that there is an ultrafilter $\mathcal{U} <_T \mathcal{U}_{top}$ [Isbell]?

2. Is Tukey equivalent to Rudin-Keisler for selective ultrafilters?

3. Is there an ultrafilter $\mathcal{U}$ on $\omega$ such that $\mathcal{U} <_T \mathcal{U} \cdot \mathcal{U} <_T \mathcal{U} \cdot \mathcal{U} \cdot \mathcal{U}$?

4. What properties are preserved Tukey downwards?