Lelek's problem is not a metric problem

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Outline

1. Two Notions
2. The Problem
3. The conversion
4. A better reflection
5. Sources

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Chainability

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$[0,1]$ is chainable; the circle S^1 is not.

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$[0, 1]$ has all spans zero, S^1 has all spans non-zero

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The problem

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Question (Lelek)

What about the converse?

This is an important problem in metric continuum theory.
We free it from the metric constraints.

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A useful tool

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Given a distributive, separative and normal latice L there is a compact Hausdorff space wL with a base for its closed sets that is isomorphic to L. wL is the Wallman space of L.

Many properties of a space X are first-order when expressed in terms of 2^X, its lattice of (all) closed sets.

Quite often, in the case of wL, it suffices to work in L only.

Reflection

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Any counterexample to Lelek's problem can be converted into a metrizable counterexample.

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Not quite . . .

Complications

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Chainability:

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where $\Phi_n(u_1, u_2, u_3, u_4)$ expresses that $\{u_1, u_2, u_3, u_4\}$ has an n-element chain refinement.

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It suffices to consider four-element open covers only.

Another complication

We have no decent formula, $L_{\omega_1,\omega}$ or otherwise, that describes in terms of 2^X that X has span (non-)zero.

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- *wL is chainable iff X is chainable*

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Theorem

In this situation:

- *wL is chainable iff X is chainable*
- *wL has span zero iff X has span zero (any kind)*

Proof for Chainability

Chainability is now first-order; we can quantify over the finite subsets of 2^X and finite ordinals.

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Furthermore, one needs only consider covers and refinements that belong to a certain base.

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For the converse . . .

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Easier said than constructed: the difficulty lies in the fact that K is not (necessarily) an elementary substructure of 2^{wK}.

Span zero, the real argument

Apply Shelah's Ultrapower theorem

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Apply Shelah's Ultrapower theorem: take a cardinal κ, an ultrafilter u on κ and an isomorphism $h : \prod_u (2^{X\times X}) \to \prod_u wK$ (which can be taken to be the identity on K).

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For that we need some topology.

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The Wallman space of the ultrapower $\prod_u B$ is the fiber $p_\kappa^\leftarrow(u)$. Bankston calls this the ultracopower of Y; we write Y_u.

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$$q_K[Z_X] = q_K\Bigl[p_{X\times X}\bigl[wh[Z_u]\bigr]\Bigr] = p_{wK}\Bigl[(wh)^{-1}\bigl[wh[Z_u]\bigr]\Bigr] = Z$$

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Finally then: if X is a non-chainable continuum that has span zero (of one of the four kinds) than so is wL.

Comment from Piotr Minc

Lelek's problem *is* a metric problem.

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Light reading

Website: `fa.its.tudelft.nl/~hart`

D. Bartošová, K. P. Hart, B. van der Steeg,
Lelek's problem is not a metric problem, to appear.