Paracompact box products then and later

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The first interesting answer: (M.E. Rudin 1972)

Theorem Assume CH. The box product of countably many compact metrizable spaces is paracompact.

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Note: Taken together these results say that something like compactness is needed, and weight must not be too large.

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I.e., forget about the box product of uncountably many spaces.

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<u>**Theorem**</u> (Wingers 1994) if $\mathfrak{d} = \mathfrak{c}$ then $\Box_{n<\omega}\Box X_n$ is paracompact if each X_n is σ-compact, 0-dimensional, first countable, $|X_n| \leq \mathfrak{c}$.

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Sketches of proofs.

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Can’t separate H, K.

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H, K can't be separated.

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Kunen also pointed out that, in the ∇-product, countable intersections (in fact $< \mathfrak{b}$ intersections) of open sets are open.

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Stratify the space.

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There's a basis of clopen sets $\{u_{x,\beta} : x \in \nabla, \beta < \mathfrak{b}\}$ where $\{u_{x,\alpha} : \alpha < \mathfrak{b}\}$ is a neighborhood base of x.

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(Hint: This is done using a scale.)

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Δ says: $\forall x$ a partial function from ω to ω $\exists f_x$ total so if $x \not\subseteq^* y$ and $y \not\subseteq^* x$ then either $\{n : y(n) \leq f_x\}$ or $\{n : x(n) \leq f_y\}$ is infinite.

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At stage α refine your cover to separate whatever is not already covered in X_α. By Δ this doesn't stop until you're finished.

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Theorem (JR) (a) If each element of $Y \subset \nabla(\omega+1)^\omega$ is "increasing", then Y is strongly closed discrete [i.e., separated by a closed discrete family.]

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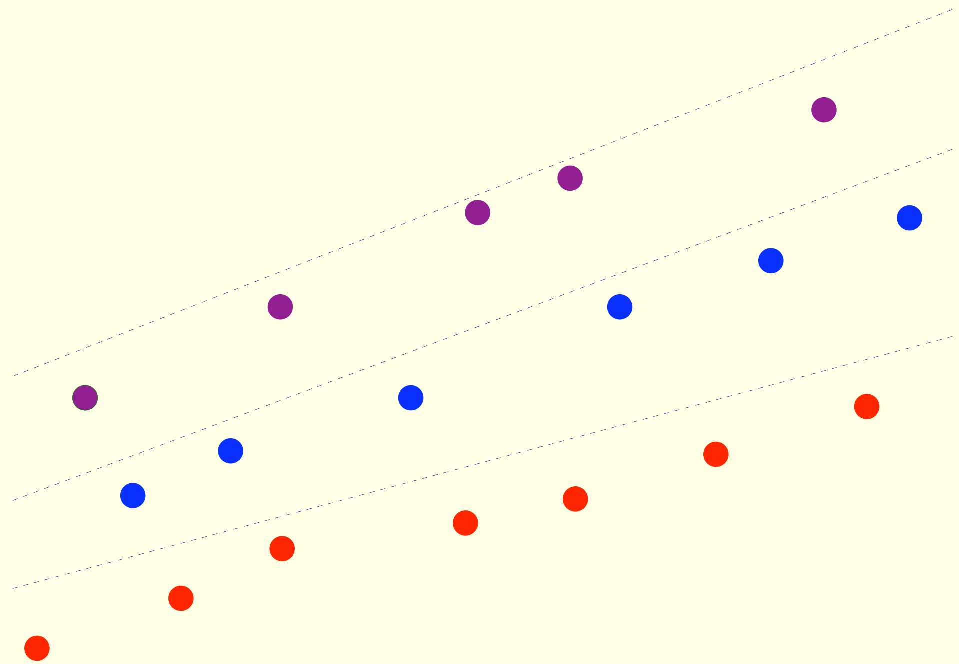
Theorem (JR) (a) If each element of $Y \subset \nabla(\omega+1)^\omega$ is "increasing", then Y is strongly closed discrete [i.e., separated by a closed discrete family.]

(b) If $\approx$ is a nice enough way of decomposing functions, and $Y \subset \nabla(\omega+1)^\omega$ is an $\approx$-transversal, then Y is paracompact.



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4. How far can you generalize counterexamples? Positive non-compact results?

5. Can you find other paracompact subspaces in ZFC?