Paracompact box products
then and later

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April 2009
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A second question: (Stone, 1950’s) Is the box product of countably many separable metric spaces normal?
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The first interesting answer: (M.E. Rudin 1972)

Theorem Assume CH. The box product of countably many compact metrizable spaces is paracompact.
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**Theorem** (a) Assume CH. If each $X_n$ is compact with each $w(X_n) \leq \omega_1$ then $\square_{n<\omega} X_n$ is paracompact.
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(b) Assume MA. If each $X_n$ is compact first countable then $\square_{n<\omega} X_n$ is paracompact.

(c) Assume CH. If each $X_n$ is compact scattered then $\square_{n<\omega} X_n$ is paracompact.
Important early negative results
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**Theorem** (Kunen 1973) If $\mathfrak{b} = \mathfrak{d}$ then $\mathfrak{d} \times \square(\omega + 1)^\omega$ is not normal.

**Theorem** (van Douwen 1975) $\mathbb{P} \times \square(\omega + 1)^\omega$ is not normal.
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**Theorem** (Kunen 1973) If $b = d$ then $d \times \square(\omega + 1)\omega$ is not normal.

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Note: Taken together these results say that something like compactness is needed, and weight must not be too large.
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I.e., forget about the box product of uncountably many spaces.
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**Theorem** (Wingers 1994) if \( d = c \) then \( \square_{n<\omega} \square X_n \) is paracompact if each 
\( X_n \) is \( \sigma \)-compact, 0-dimensional, first countable, \( |X_n| \leq c \).
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This generalizes Lawrence’s $\square \mathbb{Q}^\omega$ is paracompact.
Sketches of proofs.
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Can’t separate $H, K$. 
For $\Box (2^{(\epsilon^+)} \omega)$ to not be normal
For □(2(\(e^+\))\(\omega\)) to not be normal

Show the diagonal \(D\) is not normal.
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For \(f \in 2^A\) and \(\alpha < c^+\) define \(f_\alpha : \omega \rightarrow 2\) by \(f_\alpha(n) = f(\alpha, n)\).
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$D \cong 2^{(c^+)}$ with the $G_\delta$ topology.

$D \cong 2^A$ where $A = (c^+ \times \omega) \cup [c^+]^2$.

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$H = \{ f \in 2^A : \text{if } \alpha < \beta < c^+ \text{ and } f_\alpha = f_\beta \text{ then } f(\alpha, \beta) = 0 \}$.

$K = \{ f \in 2^A : \text{if } \alpha < \beta < c^+ \text{ then } f(\alpha, \beta) = 1 \}$.

$H, K$ can’t be separated.
Kunen’s simplifications: (good for positive results)
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**Theorem** If each $X_n$ is compact, then $\square_{n<\omega} X_n$ is paracompact iff $\nabla_{n<\omega} X_n$ is ultraparacompact.
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**Definition** Ultraparacompact: every open cover has a pairwise disjoint covering refinement.
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**Theorem** If each $X_n$ is compact, then $\Box_{n<\omega}X_n$ is paracompact iff $\nabla_{n<\omega}X_n$ is ultraparacompact.

**Definition** Ultra-paracompact: every open cover has a pairwise disjoint covering refinement.

UPC is so much nicer than PC.

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UPC is so much nicer than PC.

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Kunen also pointed out that, in the $\nabla$-product, countable intersections (in fact $< b$ intersections) of open sets are open.
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Method 2.
Stratify the space.
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There’s a basis of clopen sets $\left\{ u_{x,\beta} : x \in \nabla, \beta < b \right\}$ where $\left\{ u_{x,\alpha} : \alpha < b \right\}$ is a neighborhood base of $x$. 
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and if \( y \in u_{x,\alpha} \) and \( \beta > \alpha \) then \( u_{y,\beta} \subset u_{x,\alpha} \).
For $b = d \Rightarrow$ the box product of countably many compact metrizable spaces is paracompact

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(Hint: This is done using a scale.)
Theorem (JR) \( \Delta \Rightarrow \nabla (\omega + 1)^\omega \) is UPC.
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\( \Delta \) says: \( \forall x \) a partial function from \( \omega \) to \( \omega \) \( \exists f_x \) total so if \( x \not\sqsubset^* y \) and \( y \not\sqsubset^* x \) then either \( \{n : y(n) \leq f_x\} \) or \( \{n : x(n) \leq f_y\} \) is infinite.

Sketch of proof:
**Theorem (JR)** \( \Delta \Rightarrow \nabla (\omega + 1)^\omega \) is UPC.

\( \Delta \) says: \( \forall x \) a partial function from \( \omega \) to \( \omega \) \( \exists f_x \) total so if \( x \nsubseteq^* y \) and \( y \nsubseteq^* x \) then either \( \{ n : y(n) \leq f_x \} \) or \( \{ n : x(n) \leq f_y \} \) is infinite.

**Sketch of proof:**

Suppose \( \{ x_\alpha : \alpha < c \} \) satisfies: \( \forall x \) a partial function \( \exists \alpha \) \( x \subseteq^* x_\alpha \).
Theorem (JR) $\Delta \Rightarrow \nabla (\omega + 1)\omega$ is UPC.

$\Delta$ says: $\forall x$ a partial function from $\omega$ to $\omega$ $\exists f_x$ total so if $x \nsubseteq^* y$ and $y \nsubseteq^* x$ then either $\{n : y(n) \leq f_x\}$ or $\{n : x(n) \leq f_y\}$ is infinite.

Sketch of proof:

Suppose $\{x_\alpha : \alpha < c\}$ satisfies: $\forall x$ a partial function $\exists \alpha x \subseteq^* x_\alpha$.

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**Theorem (JR)** \( \Delta \Rightarrow \nabla (\omega + 1)^\omega \) is UPC.

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Each \( X_\alpha \) is closed discrete (in fact stronger).
Theorem (JR) $\Delta \Rightarrow \nabla (\omega + 1)^\omega$ is UPC.

$\Delta$ says: $\forall x$ a partial function from $\omega$ to $\omega\ \exists f_x$ total so if $x \not\subseteq^* y$ and $y \not\subseteq^* x$ then either $\{n : y(n) \leq f_x\}$ or $\{n : x(n) \leq f_y\}$ is infinite.

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At stage $\alpha$ refine your cover to separate whatever is not already covered in $X_\alpha$. By $\Delta$ this doesn’t stop until you’re finished.
Positive ZFC results on subsets
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**Theorem** If $|Y| \leq \mathfrak{d}$, $Y \subset \nabla_{i<\omega}$, each $X_i$ compact first countable, then $Y$ is paracompact.
Positive ZFC results on subsets

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Much more can be said about $\nabla(\omega + 1)^\omega$, e.g.,
Positive ZFC results on subsets

**Theorem** If $|Y| \leq \mathfrak{d}$, $Y \subset \nabla_{i<\omega}$, each $X_i$ compact first countable, then $Y$ is paracompact.

Much more can be said about $\nabla(\omega + 1)^\omega$, e.g.,

**Theorem** (JR) (a) If each element of $Y \subset \nabla(\omega + 1)^\omega$ is “increasing”, then $Y$ is strongly closed discrete [i.e., separated by a closed discrete family.]
Positive ZFC results on subsets

**Theorem** If $|Y| \leq \omega$, $Y \subset \nabla_{i<\omega}$, each $X_i$ compact first countable, then $Y$ is paracompact.

Much more can be said about $\nabla(\omega + 1)^\omega$, e.g.,

**Theorem** (JR) (a) If each element of $Y \subset \nabla(\omega + 1)^\omega$ is “increasing”, then $Y$ is strongly closed discrete [i.e., separated by a closed discrete family.]

(b) If $\approx$ is a nice enough way of decomposing functions, and $Y \subset \nabla(\omega + 1)^\omega$ is an $\approx$-transversal, then $Y$ is paracompact.
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4. How far can you generalize counterexamples? Positive non-compact results?
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1. Can $\Delta$ fail?

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3. Does $\Box$ normal imply $\Box$ paracompact?

4. How far can you generalize counterexamples? Positive non-compact results?

5. Can you find other paracompact subspaces in ZFC?