





K. Kunen and F. D. Tall: joint papers

-  S. BROVERMAN, J. GINSBURG, K. KUNEN, F. D. TALL
Topologies determined by σ -ideals on ω_1 .
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-  K. KUNEN, F. D. TALL
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Lindelöf spaces and selection principles

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Classic Open Questions

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4. (Michael) Is there a productively Lindelöf X such that X^ω is not Lindelöf?
5. (van Douwen) Is there a Lindelöf space which is not D ?

Definitions

- A space X has the *Rothberger (Menger)* property if for each sequence $\{\mathcal{U}_n : n < \omega\}$ of open covers of X (each closed under finite unions), for each n there is a $U_n \in \mathcal{U}_n$ such that $\{U_n : n < \omega\}$ covers X .

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- A space X is *D* if for each open neighborhood assignment $\{V_x : x \in X\}$ there is a closed discrete D such that $\{V_x : x \in D\}$ covers X .

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- A space is *indestructibly (productively) Lindelöf* if it remains (productively) Lindelöf in any countably closed forcing extension.

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Theorem 7. Every completely metrizable productively Lindelöf space is Menger (Alster) (σ -compact) (indestructibly productively Lindelöf) iff there is a Lindelöf regular space M such that $M \times \mathbb{P}$ (\mathbb{P} is the space of irrationals) is not Lindelöf.

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Corollary 8. $\mathfrak{b} = \aleph_1$ or $\mathfrak{d} = \text{cov}(\mathcal{M})$ implies every completely metrizable productively Lindelöf space is σ -compact.

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Theorem 11. *MA* implies productively Lindelöf metrizable spaces are Hurewicz.

What are the cardinalities of Rothberger spaces with points G_δ ?

Moore's L -space has cardinality \aleph_1 and is Rothberger. Gorelic's consistent Lindelöf space with points G_δ of size 2^{\aleph_1} is Rothberger.

Theorem 1. Rothberger spaces with points G_δ have cardinality $<$ the first real-valued measurable cardinal.

Proof. For each $x \in X$, choose $U_n(x)$ open, $U_n(x) \subseteq U_{n+1}(x)$, $\bigcap \{U_n(x) : n < \omega\} = \{x\}$. Take a subset Y of X of real-valued measurable size. Take $n(m, x)$ with $\mu(Y \cap U_{n(m, x)}(x)) < \frac{1}{2^{m+1}}$. Let $\mathcal{U}_m = \{U_{n(m, x)}(x) : x \in X\}$. Then, picking $U_m \in \mathcal{U}_m$, $\mu(\bigcup \{U_m : m < \omega\}) \leq \frac{1}{2}$, so can't cover Y , which has measure 1, so can't cover X .

Ideas of Proofs

Theorem 2. If it is consistent there is a supercompact cardinal, it is consistent with *GCH* that all Rothberger spaces with points G_δ have cardinality $\leq \aleph_1$, and that all uncountable Rothberger spaces of character $\leq \aleph_1$ have Rothberger subspaces of size \aleph_1 .

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- a) Rothberger spaces are indestructible. [Tall] proved: Lévy collapse a supercompact to ω_2 . Then indestructibly Lindelöf spaces with points G_δ have cardinality $\leq \aleph_1$.

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- b) In the same model, by the character restriction, $j''X$ is a subspace of $j(X)$. By indestructibility, it is Rothberger. Then $j(X)$ has a Rothberger subspace of size $\aleph_1 = j(\aleph_1)$, so by elementarity, so does X .

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Lemma [Alster]. $CH \rightarrow$ productively Lindelöf spaces with $w \leq \aleph_1$ are Alster.

Collapse $\max(2^{\aleph_0}, w(X))$ to \aleph_1 and apply Lemma.

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Lemma. Alster \rightarrow Hurewicz.

Proof. Use characterization of Hurewicz: For each Čech-complete $G \supseteq X$, there is a σ -compact F such that $X \subseteq F \subseteq G$.

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Lemma [Alster]. X Alster implies X^ω is Lindelöf.

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Lemma [Lawrence]. Suppose Y is a separable completely metrizable space which is not σ -compact. Then $X \times Y$ is normal if and only if $X \times \mathbb{P}$ is normal.

Lemma [Alster]. For metrizable spaces, Alster = σ -compact.

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To prove this, divide into cases.

Case 1: $\mathfrak{d} = \aleph_1$. Then proof of Alster's Lemma works.

Case 2: $\mathfrak{d} > \aleph_1$. Then every separable metric space of size \aleph_1 is Menger.

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Theorem 11. *MA* implies productively Lindelöf metrizable spaces are Hurewicz.

Proof. Follows from work of Alster re Michael spaces, plus characterization of Hurewicz.

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D -spaces, topological games and selection principles.

Submitted.



AURICHI, L. F. AND TALL, F. D.

Lindelöf spaces which are indestructible, productive, or D .

Submitted.



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SCHEEPERS, M., AND TALL, F. D.

Lindelöf indestructibility, topological games and selection principles.

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TALL, F. D.

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References III



TALL, F. D.

Lindelöf spaces which are productive, Alster, Hurewicz, Menger, or D .

In preparation.

In the following diagram, a solid arrow with a number n next to it indicates that Example n is a counterexample to the reverse arrow; a broken dashed arrow with a number m next to it indicates that Example m shows the implication does not hold; a dotted arrow indicates the implication holds, if the additional hypothesis next to it is assumed.

Key to arc labels

1. Moore's L -space.
2. $[0, 1]$.
3. The irrationals.
4. 2^{ω_1} .
5. A Hurewicz (and hence Menger) subspace of the real line which is not σ -compact (and hence not Alster).
6. Michael's space.
7. The one-point Lindelöfication of the discrete space of size \aleph_1 .
8. A Bernstein set.
9. Przymusiński's space.
10. Another example of Przymusiński.
11. The Sorgenfrey line ($(\text{perfect} + \text{Lindelöf}) \times \text{separable Lindelöf}$ is Lindelöf).
12. A Menger subspace of the real line which is not Hurewicz.
13. The subspace of the Michael line obtained from a set concentrated on the rationals.