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Lindelöf spaces and selection principles

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Classic Open Questions

We report on recent research in collaboration with Marion Scheepers and with Leandro Aurich. Classical combinatorial strengthenings of Lindelöfness, namely the Menger and Rothberger properties, yield new insights into longstanding open problems in topology.
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5. (van Douwen) Is there a Lindelöf space which is not $D$?
A space $X$ has the Rothberger (Menger) property if for each sequence $\{U_n : n < \omega\}$ of open covers of $X$ (each closed under finite unions), for each $n$ there is a $U_n \in U_n$ such that $\{U_n : n < \omega\}$ covers $X$. 
Definitions

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- A space $X$ has the **Hurewicz** property if for each sequence $\{U_n : n < \omega\}$ of open covers of $X$, there is a sequence $\{V_n : n < \omega\}$ of finite sets such that $V_n \subseteq U_n$, and each $x \in X$ is in $\bigcup V_n$ for all but finitely many $n$. 
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• A space $X$ is $D$ if for each open neighborhood assignment $\{V_x : x \in X\}$ there is a closed discrete $D$ such that $\{V_x : x \in D\}$ covers $X$. 
Definitions

• A space is *Alster* if every cover by $G_\delta$ sets that covers each compact set finitely includes a countable subcover.
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- A space is *Alster* if every cover by $G_\delta$ sets that covers each compact set finitely includes a countable subcover.
- A space $X$ is *productively Lindelöf* if $X \times Y$ is Lindelöf for every Lindelöf space $Y$.
- A space is *indestructibly (productively) Lindelöf* if it remains (productively) Lindelöf in any countably closed forcing extension.
Theorem 1 [Scheepers-Tall]. Rothberger spaces with points $G_\delta$ have cardinality $< \text{the first real-valued measurable cardinal}$. 
Theorems

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**Theorem 2 [Scheepers-Tall].** If it is consistent there is a supercompact cardinal, it is consistent with $GCH$ that all Rothberger spaces with points $G_\delta$ have cardinality $\leq \aleph_1$, and that all uncountable Rothberger spaces of character $\leq \aleph_1$ have Rothberger subspaces of size $\aleph_1$. 
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Theorem 5. Alster implies Hurewicz implies Menger.
Theorem 6. \( CH \) implies that if a \( T_3 \) space \( X \) is either separable or first countable, and is productively Lindelöf, then it is Alster and hence Hurewicz, Menger, and \( D \).
Theorems

**Theorem 6.** *CH* implies that if a $T_3$ space $X$ is either separable or first countable, and is productively Lindelöf, then it is Alster and hence Hurewicz, Menger, and $D$.

**Theorem 7.** Every completely metrizable productively Lindelöf space is Menger (Alster) ($\sigma$-compact) (indestructibly productively Lindelöf) iff there is a Lindelöf regular space $M$ such that $M \times \mathbb{P}$ ($\mathbb{P}$ is the space of irrationals) is not Lindelöf.
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**Corollary 8.**  \( b = \aleph_1 \) or \( d = \text{cov}(M) \) implies every completely metrizable productively Lindelöf space is \( \sigma \)-compact.
Theorem 9. \( \mathfrak{c} = \aleph_1 \) implies every productively Lindelöf metrizable space is \( \sigma \)-compact.
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**Theorem 10.** Productively Lindelöf metrizable spaces of size \( \aleph_1 \) are Menger.
Theorems

**Theorem 9.** \( d = \aleph_1 \) implies every productively Lindelöf metrizable space is \( \sigma \)-compact.

**Theorem 10.** Productively Lindelöf metrizable spaces of size \( \aleph_1 \) are Menger.

**Theorem 11.** MA implies productively Lindelöf metrizable spaces are Hurewicz.
What are the cardinalities of Rothberger spaces with points $G_\delta$?

Moore’s $L$-space has cardinality $\aleph_1$ and is Rothberger. Gorelic’s consistent Lindelöf space with points $G_\delta$ of size $2^{\aleph_1}$ is Rothberger.

**Theorem 1.** Rothberger spaces with points $G_\delta$ have cardinality $<\text{the first real-valued measurable cardinal.}$

**Proof.** For each $x \in X$, choose $U_n(x)$ open, $U_n(x) \subseteq U_{n+1}(x)$, $\bigcap\{U_n(x) : n < \omega\} = \{x\}$. Take a subset $Y$ of $X$ of real-valued measurable size. Take $n(m, x)$ with $\mu(Y \cap U_{n(m,x)}(x)) < \frac{1}{2^{m+1}}$. Let $U_m = \{U_{n(m,x)}(x) : x \in X\}$. Then, picking $U_m \in U_m$, $\mu(\bigcup\{U_m : m < \omega\} \leq \frac{1}{2})$, so can’t cover $Y$, which has measure 1, so can’t cover $X$. 
Ideas of Proofs

**Theorem 2.** If it is consistent there is a supercompact cardinal, it is consistent with $GCH$ that all Rothberger spaces with points $G_\delta$ have cardinality $\leq \aleph_1$, and that all uncountable Rothberger spaces of character $\leq \aleph_1$ have Rothberger subspaces of size $\aleph_1$. 
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Proof.

a) Rothberger spaces are indestructible. [Tall] proved: Lévy collapse a supercompact to $\omega_2$. Then indestructibly Lindelöf spaces with points $G_\delta$ have cardinality $\leq \aleph_1$. 
Theorem 2. If it is consistent there is a supercompact cardinal, it is consistent with GCH that all Rothberger spaces with points $G_\delta$ have cardinality $\leq \aleph_1$, and that all uncountable Rothberger spaces of character $\leq \aleph_1$ have Rothberger subspaces of size $\aleph_1$.

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b) In the same model, by the character restriction, $j''X$ is a subspace of $j(X)$. By indestructibility, it is Rothberger. Then $j(X)$ has a Rothberger subspace of size $\aleph_1 = j(\aleph_1)$, so by elementarity, so does $X$. 
Ideas of Proofs

**Theorem 3.** Menger spaces are $D$-spaces.

**Theorem 4.** Indestructibly productively Lindelöf implies Alster implies Hurewicz implies Menger.
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**Proof.** Aurichi: Menger $\rightarrow D$ via topological games.
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Lemma [Alster]. $CH \rightarrow$ productively Lindelöf spaces with $w \leq \aleph_1$ are Alster.

Collapse $\max(2^\aleph_0, w(X))$ to $\aleph_1$ and apply Lemma.
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**Lemma (classic).** Hurewicz $\rightarrow$ Menger.

**Lemma.** Alster $\rightarrow$ Hurewicz.

**Proof.** Use characterization of Hurewicz: For each Čech-complete $G \supseteq X$, there is a $\sigma$-compact $F$ such that $X \subseteq F \subseteq G$. 
**Theorem 6.**  \( CH \) implies that if a \( T_3 \) space \( X \) is either separable or first countable, and is productively Lindelöf, then it is Alster and hence Menger and \( D \).
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Proof.  Note separable regular spaces and first countable Lindelöf Hausdorff spaces have \( w \leq 2^\aleph_0 \) and apply Alster’s Lemma.
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Proof. Note separable regular spaces and first countable Lindelöf Hausdorff spaces have \( w \leq 2^\aleph_0 \) and apply Alster’s Lemma.

Lemma [Alster]. \( X \) Alster implies \( X^\omega \) is Lindelöf.
Theorem 7 (there is a Michael space iff completely metrizable productively Lindelöf spaces are $\sigma$-compact) follows from:
Ideas of Proofs

Theorem 7 (there is a Michael space iff completely metrizable productively Lindelöf spaces are \( \sigma \)-compact) follows from: Lemma [Rudin-Starbird]. Suppose \( X \) is Lindelöf regular and \( Y \) is separable metrizable. Then \( X \times Y \) is normal if and only if \( X \times Y \) is Lindelöf.
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**Lemma [Rudin-Starbird].** Suppose $X$ is Lindelöf regular and $Y$ is separable metrizable. Then $X \times Y$ is normal if and only if $X \times Y$ is Lindelöf.

**Lemma [Lawrence].** Suppose $Y$ is a separable completely metrizable space which is not $\sigma$-compact. Then $X \times Y$ is normal if and only if $X \times \mathbb{P}$ is normal.
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Lemma [Alster]. For metrizable spaces, Alster $= \sigma$-compact.
Theorem 10. Productively Lindelöf metrizable spaces of size $\leq \aleph_1$ are Menger.
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Case 1: $\emptyset = \aleph_1$. Then proof of Alster’s Lemma works.
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**Theorem 10.** Productively Lindelöf metrizable spaces of size \( \leq \aleph_1 \) are Menger.

To prove this, divide into cases.

Case 1: \( d = \aleph_1 \). Then proof of Alster’s Lemma works.

Case 2: \( d > \aleph_1 \). Then every separable metric space of size \( \aleph_1 \) is Menger.
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**Theorem 10.** Productively Lindelöf metrizable spaces of size \( \leq \aleph_1 \) are Menger.

To prove this, divide into cases.

Case 1: \( \mathfrak{d} = \aleph_1 \). Then proof of Alster’s Lemma works.

Case 2: \( \mathfrak{d} > \aleph_1 \). Then every separable metric space of size \( \aleph_1 \) is Menger.

**Theorem 11.** \( \text{MA} \) implies productively Lindelöf metrizable spaces are Hurewicz.

**Proof.** Follows from work of Alster re Michael spaces, plus characterization of Hurewicz.
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Lindelöf spaces which are productive, Alster, Hurewicz, Menger, or $D$.

In preparation.
In the following diagram, a solid arrow with a number $n$ next to it indicates that Example $n$ is a counterexample to the reverse arrow; a broken dashed arrow with a number $m$ next to it indicates that Example $m$ shows the implication does not hold; a dotted arrow indicates the implication holds, if the additional hypothesis next to it is assumed.
Key to arc labels

1. Moore’s $L$-space.
2. $[0, 1]$.
3. The irrationals.
4. $2^\omega_1$.
5. A Hurewicz (and hence Menger) subspace of the real line which is not $\sigma$-compact (and hence not Alster).
6. Michael’s space.
7. The one-point Lindelöfication of the discrete space of size $\aleph_1$.
9. Przymusiński’s space.
10. Another example of Przymunski’s.
11. The Sorgenfrey line (perfect + Lindelöf $\times$ separable Lindelöf is Lindelöf).
12. A Menger subspace of the real line which is not Hurewicz.
13. The subspace of the Michael line obtained from a set concentrated on the rationals.