

# Tall PCF spaces

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# Outline

- 1 Cardinal arithmetic
- 2 PCF theory
- 3 PCF structures
- 4 LCS spaces

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# Cardinal arithmetic

Cardinal arithmetic reduces to the study of cardinal exponentiation. In fact, it suffices to consider the function  $\kappa \mapsto \kappa^{\text{cof}(\kappa)}$ .

In the case of regular cardinals, the only limitation is the following.

## Theorem (König)

*If  $\kappa$  is an infinite cardinal then  $\kappa^{\text{cof}(\kappa)} > \kappa$ . In particular,  $\text{cof}(2^\kappa) > \kappa$ , for all infinite  $\kappa$ .*

## Theorem (Easton)

*Let  $F : \text{RCard} \rightarrow \text{Card}$  be a class function, where  $\text{Card}$  is the class of infinite cardinals and  $\text{RCard}$  is the class of regular cardinals. Assume that*

- $\kappa \leq \lambda \rightarrow F(\kappa) \leq F(\lambda)$
- $\text{cof}(F(\kappa)) > \kappa$ , for all regular  $\kappa$ .

*Then there is a generic extension of the universe in which  $2^\kappa = F(\kappa)$ , for all regular  $\kappa$ .*

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# Singular Cardinals

Concerning singular cardinals there are several restriction on the value of the exponential function.

## Theorem (Silver, 1974)

*Assume  $\kappa$  is a singular cardinal of uncountable cofinality and  $2^\delta = \delta^+$ , for all  $\delta < \kappa$ . Then  $2^\kappa = \kappa^+$ .*

## Theorem (Galvin, Hajnal, 1975)

*Assume  $\aleph_\delta$  is a singular strong limit cardinal of uncountable cofinality. Then  $2^{\aleph_\delta} < \aleph_{(|\delta| \text{cof}(\delta))^+}$ .*

## Theorem (Shelah, 1978)

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# Shelah's theorem

Starting from the early 1980s Shelah developed PCF theory which allowed him to obtain much stronger restrictions on the power of singular cardinals. The most spectacular application of this theory is the following.

**Theorem (Shelah, 1989)**

$$\aleph_\omega^{\aleph_0} < \max(\aleph_{\omega_4}, (2^{\aleph_0})^+).$$

**Corollary**

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# Singular Cardinal Hypothesis

In view of the previous results one may ask what are the restrictions on the exponential function on singular cardinals. The **Singular Cardinal Hypothesis** says that it takes the minimal possible value.

## Definition

The Singular Cardinal Hypothesis **SCH** says that, for every singular cardinal  $\kappa$ , if  $2^{\text{cof}(\kappa)} < \kappa$  then  $\kappa^{\text{cof}(\kappa)} = \kappa^+$ .

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- Solovay showed SCH holds above a strongly compact cardinal.
- Viale showed that SCH follows from the Proper Forcing Axiom.
- Using a supercompact cardinal Silver showed that SCH can consistently fail.
- Magidor showed, using large cardinals, that SCH can first fail at  $\aleph_\omega$ . The consistency strength was later reduced by Gitik and Magidor.
- Shelah showed using large cardinals that for any given countable ordinal  $\alpha$  it is possible to have  $\aleph_\omega$  strong limit and  $2^{\aleph_\omega} = \aleph_{\alpha+1}$ .
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# Definitions

- If  $A$  is a set of regular cardinals we let  $\prod A$  denote the set of all functions  $f$  with domain  $A$  such that  $f(a) \in a$ , for all  $a \in A$ .
- If  $F$  is a filter on  $A$  we define a partial ordering  $\leq_F$  on  $\prod A$  by  $f \leq_F g$  iff  $\{a \in A : f(a) \leq g(a)\} \in F$ .
- The **cofinality** of  $\prod A/F$  is the least size of a set cofinal in  $(\prod A, \leq_F)$ .
- $\text{pcf}(A) = \{\text{cof}(\prod A, \leq_D) : D \text{ is an ultrafilter on } A\}$ .
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### **Theorem (Shelah)**

*Let  $A$  be a progressive interval of regular cardinals. Then  $\text{pcf}(A)$  is also an interval of regular cardinals and has a maximal element.*

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*Suppose  $\mu$  is a singular cardinal and  $\kappa < \mu$  is an infinite cardinal such that the interval  $A$  of regular cardinals in  $(\kappa, \mu)$  has cardinality  $\leq \kappa$ . Then*

$$\text{cof}([\mu]^\kappa, \subseteq) = \max(\text{pcf}(A)).$$

### **Corollary**

*Under the same assumptions  $\mu^\kappa = \max\{2^\kappa, \max(\text{pcf}(A))\}$ .*

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# Properties of the PCF space

Let  $A$  be a progressive interval of regular cardinals. We can consider  $A$  and  $\text{pcf}(A)$  as sets of ordinals (to  $\alpha$  we associate  $\aleph_{\alpha+1}$ ). If  $\rho = \max(\text{pcf}(A))$  then the pcf operator on  $I = [\min(A), \rho]$  has the following properties:

- If  $X, Y \subseteq I$  then  $X \subseteq \text{pcf}(X)$ ,  $\text{pcf}(X \cup Y) = \text{pcf}(X) \cup \text{pcf}(Y)$ ,  $\text{pcf}(\text{pcf}(X)) = \text{pcf}(X)$ .
- If  $X \subseteq I$  and  $\gamma \in \text{pcf}(X)$  then there is  $Y \subseteq X$  such that  $\gamma \in \text{pcf}(Y)$  and  $|Y| \leq |A|$ .
- For every nonempty  $X \subseteq I$ ,  $\text{pcf}(X)$  has a maximal element and  $\min(\text{pcf}(X)) = \min(X)$ .
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Thus, we obtain a topological space on  $I$  with some very special properties.  $A$  is a dense subset of  $I$  in this topology and there is an interplay between the natural ordering of ordinals and the topological properties of the space.

### Theorem (Shelah)

*Let  $A$  and  $I$  be intervals of ordinals with  $A$  an initial segment of  $I$ . Assume there is a closure operation on  $I$  with the above properties. Then  $|I| < |A|^{+4}$ .*

### Corollary

*Let  $A$  be a progressive interval of regular cardinals. Then  $|\text{pcf}(A)| < |A|^{+4}$ .*

### The PCF conjecture

*If  $A$  is a progressive interval of regular cardinals then  $|\text{pcf}(A)| \leq |A|$ .*

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*Let  $A$  be a progressive interval of regular cardinals. Then  $|\text{pcf}(A)| < |A|^{+4}$ .*

### **The PCF conjecture**

If  $A$  is a progressive interval of regular cardinals then  $|\text{pcf}(A)| \leq |A|$ .

# Outline

- 1 Cardinal arithmetic
- 2 PCF theory
- 3 PCF structures**
- 4 LCS spaces

# PCF structures

## Definition

A topological space  $X$  is called **scattered** if every nonempty subset of  $X$  has an isolated point.

## Definition (Cantor Bendixson derivative)

Let  $X$  be a topological space. The  $\alpha$ -th derivative  $X^{(\alpha)}$  of  $X$  is defined as follows:  $X^{(0)} = X$ ,  $X^{(\alpha+1)} = X^{(\alpha)} \setminus I(X^{(\alpha)})$ , where  $I(Y)$  denotes the set of isolated points of  $Y$ . If  $\alpha$  is a limit ordinal then  $X^{(\alpha)} = \bigcap_{\xi < \alpha} X^{(\xi)}$ .

- If  $X$  is scattered then there is  $\alpha$  such that  $X^{(\alpha)} = \emptyset$ . The **height** of  $X$  is the least such  $\alpha$ .
- We call  $I(X^{(\alpha)})$  the  **$\alpha$ -th level** of  $X$ .
- The **width** of  $X$  is the supremum of the cardinality of the levels.

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Let  $X$  be a topological space. The **tightness** of  $X$  is the least cardinal  $\kappa$  such that if  $Y \subseteq X$  and  $p \in \bar{Y}$  there is  $Z \subseteq Y$  with  $|Z| \leq \kappa$  such that  $p \in \bar{Z}$ .

We saw that if  $A$  is a progressive interval of regular cardinals then the pcf space on  $\text{pcf}(A)$  has tightness at most  $|A|$ .

## Definition

Let  $I$  be an interval of ordinals and  $\tau$  a topology on  $I$  which refines the order topology. We say that  $(I, \tau)$  has the **club property** if for every ordinal  $\alpha \in I$  of uncountable cofinality there is a club  $C \subseteq \alpha$  such that  $\bar{C} \subseteq \alpha + 1$ .

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Let  $\Theta$  be an ordinal and  $\tau$  a topology on  $\Theta$  refining the order topology. We say that  $(\Theta, \tau)$  is a **pcf candidate** if it is locally compact, countably tight and has the club property.

We saw that if  $A = \{\aleph_n : n < \omega\}$  and  $\max(\text{pcf}(A)) = \aleph_{\Theta+1}$  then using the pcf operation we can define a pcf candidate  $(\Theta, \tau)$ . (Recall that we identify  $\alpha$  and  $\aleph_{\alpha+1}$ .)

## Theorem (Shelah)

*If  $(\Theta, \tau)$  is a pcf candidate then  $\Theta < \omega_4$ .*

## Question

Is this bound optimal?

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## Definition

Let  $X$  be a locally compact scattered space. Let  $\lambda$  be the height of  $X$  and let  $\kappa_\alpha$  be the cardinality of the  $\alpha$ -th level of  $X$ . Then  $\langle \kappa_\alpha : \alpha < \lambda \rangle$  is the **cardinal sequence** of  $X$ .

The question of which sequences of cardinals can occur as cardinal sequences of locally compact scattered spaces has been studied extensively. We do not give a complete survey but just mention the key results.

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The question of which sequences of cardinals can occur as cardinal sequences of locally compact scattered spaces has been studied extensively. We do not give a complete survey but just mention the key results.

Let  $\vec{\theta} = \langle \kappa_\alpha : \alpha < \lambda \rangle$ , where each  $\kappa_\alpha$  is an infinite cardinal.  $(T, \leq, i)$  is a  $\vec{\theta}$ -poset if

- ①  $T = \bigcup \{T_\alpha : \alpha < \lambda\}$ , where  $T_\alpha = \{\alpha\} \times Y_\alpha$ , and  $|Y_\alpha| = \kappa_\alpha$ .
- ②  $i : [T]^2 \rightarrow [T]^{<\omega}$  satisfies:
  - ① If  $u \in i\{s, t\}$ , then  $u \leq s, t$
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Let  $\Omega(\lambda)$  denote the constant sequence of length  $\lambda$  where each value is  $\omega$ . It is fairly easy to construct an  $\Omega(\omega_1)$ -poset in ZFC alone.

### Theorem (Baumgartner, Shelah)

*It is relatively consistent with ZFC that there exist an  $\Omega(\omega_2)$ -poset.*

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*It is relatively consistent with ZFC that there exist an  $\Omega(\delta)$ -poset, for all  $\delta < \omega_3$ .*

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# Tall pcf spaces

**Question:** Can we use the above constructions to obtain tall pcf candidates? We need to add the club property.

**Answer:** Yes, but there are technical difficulties.

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Let  $f : [\omega_2]^2 \rightarrow [\omega_2]^\omega$  be such that  $f\{\alpha, \beta\} \subseteq \alpha \cap \beta$ , for all  $\alpha, \beta$ . We say that two finite subsets  $x$  and  $y$  of  $\omega_2$  are **good** for  $f$  if for all  $\tau \in x \cap y$ ,  $\alpha \in x \setminus y$  and  $\beta \in y \setminus x$

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Baumgartner and Shelah first add a  $\Delta$ -function by forcing and then use it to force an  $\Omega(\omega_2)$ -poset. It is possible to obtain a  $\Delta$ -function directly from  $\square_{\omega_1}$ .

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Can one build pcf candidates of height  $\delta$ , for any  $\delta < \omega_4$ ?

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