Metric Continuous Abstract Elementary Classes

Why? Superstability, NIP, examples.

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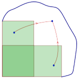
Universidad Nacional - Bogotá

KunenFest - Madison - April 2009



Contents

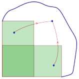
Goals + Acknowledgments

Various notions at the crossroads of Combinatorial Set Theory and Model Theory. . .

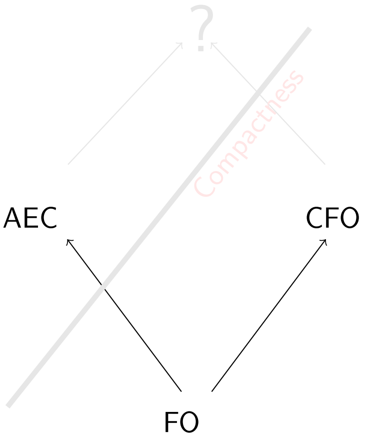
Madison connection : Kunen - Keisler [Enayat, Džamonja]

I will put in context results or constructions due to Berenstein, Grossberg, Hyttinen, Shelah, VanDieren, V., Zambrano on Continuous Model Theory, and Metric Continuous Abstract Elementary Classes (mcAEC).

. . . and several open problems.



Beyond FO : two ways (analysis or algebra)



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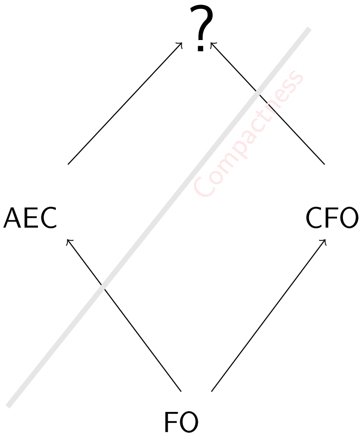

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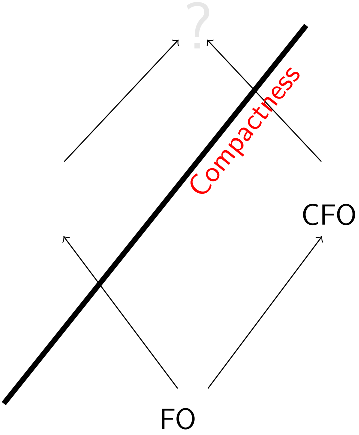
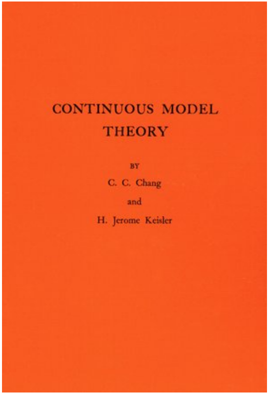


Continuous Model Theory - Origins



Although the origins of CMTh go back to Chang & Keisler (1966), recent takes on Continuous Model Theory are based on formulations due to Ben Yaacov, Usvyatsov and Berenstein of Henson and Iovino's Logic for Banach Spaces.

Unlike the case of AEC, these authors emphasize control of structures through formulas (and *syntactic* types).

The context obtained is (up to some point) "orthogonal" to AEC : while AEC does not capture the notion of approximation (Banach Spaces do not form an AEC !), all results in CMTh (so far) depend on strong compactness and countability (separability) hypotheses.

Continuous predicates and functions

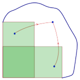
Definition

Fix (M, d) a bounded metric space. A continuous n-ary predicate is a uniformly continuous function

$$P : M^n \longrightarrow [0, 1].$$

A continuous n-ary function is a uniformly continuous function

$$f : M^n \longrightarrow M.$$



Metric structures

Therefore, metric structures are of the form

$$\mathcal{M} = \Big(M, d, (f_i)_{i \in I}, (R_j)_{j \in J}, (a_k)_{k \in K}\Big)$$

where the R_i and the f_j are (uniformly) continuous functions with values in $[0,1]$, the a_k are distinguished elements of M.
Remember : M is a bounded metric space.
Each function, relation must be endowed with a modulus of uniform continuity.

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Examples of FO metric structures

Example

- Any FO structure, endowed with the discrete metric.
- The unit ball B of a normed space X over $\mathbb{R}$ or $\mathbb{C}$ - functions : maps $(x, y) \mapsto \alpha x + \beta y$ for each pair of scalars such that $|\alpha| + |\beta| \leq 1$. Norm included as predicate.
- Normed algebras.
- Hilbert spaces with inner product as a binary predicate.
- For a probability space $(\Omega, \mathcal{B}, \mu)$, construct a metric structure $\mathcal{M}$ based on the usual measure algebra of $(\Omega, \mathcal{B}, \mu)$. Operations : $\cap, \cup$, complement. Predicate : measure μ. Distinguished elements : $0, 1$.
- C^*-algebras (Argoty, Ben Yaacov, V.).

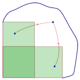
Some of the goods

- Stability (Ben Yaacov, Iovino, etc.),
- Ryll-Nardzewski (Berenstein),
- Categoricity for countable languages (Ben Yaacov),
- ω-stability,
- Dependent theories (Ben Yaacov),
- Not much geometric stability theory : no analog to Baldwin-Lachlan (no minimality),
- NO simplicity ! ! !
- Keisler measures, NIP (Hrushovski, Pillay, etc.)

mcAEC

A class of L-(metric) structures, $(\mathcal{K}, \prec_{\mathcal{K}})$ (where predicates in L are $[0,1]$-valued and continuous, functions are continuous), is said to be a metric continuous abstract elementary class (mcAEC) if both $\mathcal{K}$ and the binary relation $\mathcal{K}$ are closed under isomorphism and

- T-V $A \prec_{\mathcal{K}} C, B \prec_{\mathcal{K}} C$ and $A \subset B$ then $A \prec_{\mathcal{K}} B$.
- Lim Closed under completions of direct limits of $\prec_{\mathcal{K}}$-chains.
- Con Smoothness axiom (for completions).
- $\searrow$LS Downward Löwenheim-Skolem for density character.



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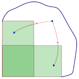

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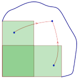

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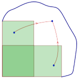


mcAEC and AEC - Jónsson AEC

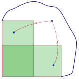
Definition (or Remark)

When the metric is discrete, mcAECs are just regular AECs.

So, mcAEC generalize in a natural way both AEC and classes axiomatizable in FO Continuous Logic.

Definition (Jónsson AEC)

An AEC that satisfies AP, JEP and has arbitrarily large models is called a Jónsson AEC.



Examples

Many natural constructions in Mathematics are examples of mcAEC

1. Complete first order theories
2. Homogeneous Model Theory
3. Excellent, quasiminimal classes
4. Various classes axiomatizable in $L_{\omega_1,\omega}$ or $L_{\kappa\omega}$.
5. Covers of Abelian algebraic groups
6. Hilbert Spaces with Annihilation and Creation operators
7. Classes associated to C^*-algebras
8. The Zilber analytic classes (pseudoexponentiation)
9. Hart-Shelah examples (Baldwin, Kolesnikov, V.)
10. Enayat-Schmerl-Visser's ω-models of T_{fin}, end extension

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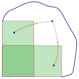

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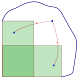
An interesting dichotomy

1. (under Weak Diamond at κ ($2^\kappa < 2^{\kappa^+}$)) Every AEC which fails AP at κ, and κ-categorical has 2^{κ^+} many non-isomorphic models of size κ^+.
2. (under MA) There exists an AEC that fails AP at $\aleph_0$, is $\aleph_0$-categorical BUT is also $\aleph_1$-categorical.



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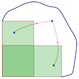
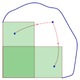


An interesting dichotomy · {notes}

The proof under $2^\kappa < 2^{\kappa^+}$ uses the weak diamond to get many different models (easier version : prove there is no universal model of size κ^+).
The construction under MA uses a variant of Baumgartner's forcing to build, given two models of size $\aleph_1$, an isomorphism. The construction is essentially a random bipartite graph, with stochastically independent families. OPEN : an example describable in $L_{\omega_1\omega}$.
OPEN : can we push up this dichotomy ? Until which cardinalities ?



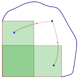
Limit models : $\exists$ and !

Theorem (Shelah)

Let $\mathcal{K}$ be a μ-Galois-stable AEC with AP and extensions of models. Then $\mathcal{K}$ has (μ, α)-limit models for all $\alpha \leq \mu$.

Theorem (Shelah, V.)

(GCH) If $\mathcal{K}$ is a λ-categorical AEC with NMM, then $\mathcal{K}$ has (μ, α)-limit models for all $\alpha \leq \mu$.

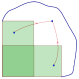


Uniqueness of limit models

Let
M_1 a (μ, α_1)-limit model over M_0 and
M_2 a (μ, α_2)-limit model over M_0.
Must they be isomorphic ? Moreover, ¿can we get

$$M_1 \approx_{M_0} M_2 \, ?$$

Under AP, they must be isomorphic, when the class is categorical above μ and has arbitrarily large models.
This uses extensively EM models and splitting.
Without AP, the answer is YES if the class if categorical above μ under NMM (Shelah, V. [ShVi 635] and VanDieren).



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Protean superstability

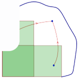
For T a complete first order theory, TFAE :

1. T is superstable,
2. $\kappa(T) = \aleph_0$,
3. Any increasing union of saturated models is saturated,
4. $n_\mu(T) = 1$ for μ regular $> |T|^+$,
5. Existence and Uniqueness of limit models of cardinality $\mu \geq |T|^+$.

Definition

Fix M a saturated model of size λ^+.

$$n_\lambda(T) := \min \left\{ |M_i/\approx| \;\middle|\; \begin{array}{l} i \in E, E \text{ club of } \lambda^+, \\ \langle M_i | i < \lambda^+ \rangle \text{ resolution of } M \end{array} \right\}$$



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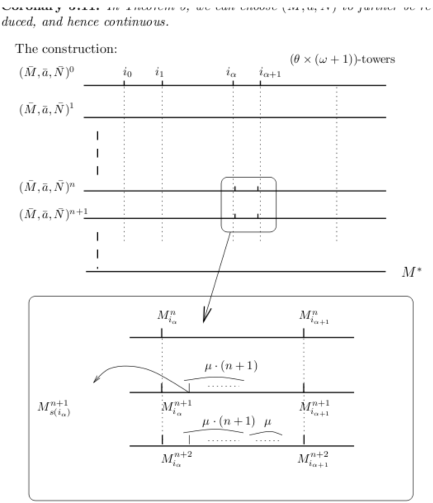
Limits and superstability

- The implication 2 $\Rightarrow$ 5 lifts to AECs (Grossberg-VanDieren-V.) : superstable AEC's have uniqueness of limit models. No EM models used ! Two-dimensional arrays of models instead of towers.
- V.-Zambrano : uniqueness of limit models for superstable mcAEC.
- Shelah uses various forms of limit models to characterize (externally) NIP theories.
- Open : behavior of $n_\mu(\mathcal{K})$.

Picture of the proof



Limit models

Limit models are connected with steps toward Shelah's Categoricity Conjecture, AEC without AP, Stability Spectrum for tame AEC (Baldwin, Kueker, VanDieren), superstability in AEC (Grossberg, VanDieren, V.) and more recently with classification of models of (First Order) NIP theories (various forms of limit models in [Sh 868] and [Sh 900]).

Keisler measures

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North-Holland

MEASURES AND FORKING

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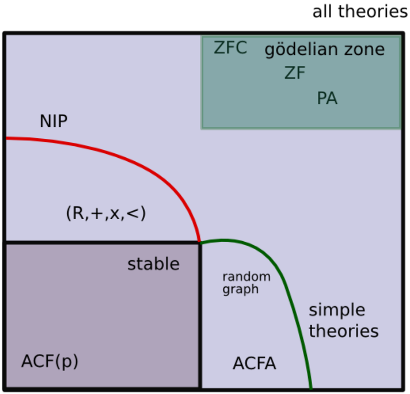
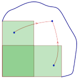
Shelah's theory of forking (or stability theory) is generalized in a way which deals with measures instead of complete types. This allows us to extend the method of forking from the class of stable theories to the larger class of theories which do not have the independence property. When restricted to the special case of stable theories, this paper reduces to a reformulation of the classical approach. However, it goes beyond the classical approach in the case of unstable theories. Methods from ordinary forking theory and the Loeb measure construction from nonstandard analysis are used.

Keisler measures (1987 !) are generalizations of types originally devised for probability logic, and recently used by Hrushovski and Pillay for NIP contexts ! They appear to point to the right notion of types for some mcAEC's (C^*-algebras).

Simple theories in the FO map



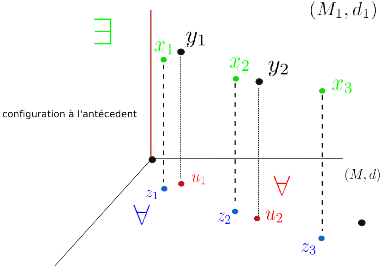


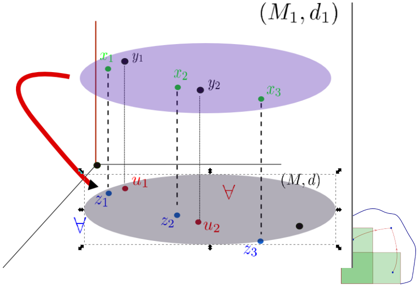
Is simplicity a discrete phenomenon ?

Theorem (Hyttinen, V.)

The (continuous) FO theory of Hilbert Spaces with Generic Predicates is not simple.

(Contrast with Chatzidakis-Pillay - Ben Yaacov's Conjecture)





The failure of simplicity

The proof's idea uses a criterion from Shelah and Casanovas : T is simple iff there exist κ, λ such that $NT(\kappa, \lambda) < 2^\kappa + \lambda^\omega$.
So, fix κ an infinite cardinal and $\lambda \geq \kappa$. The idea is to find a complete submodel M_f of the monster model, of density character λ and λ^κ many types over a subset of M_f of power κ such that they are pairwise incompatible.
This is impossible in first order, by Chatzidakis-Pillay.
The continuity of the predicate allows a situation impossible in FO.

The failure of simplicity (II)

Let then $\{b_i | i < \kappa\} \cup \{a_j | j < \lambda\} \cup \{c_X | X \in P_\kappa(\lambda)\}$ be an orthonormal basis of M_f.
Fix, for every $X \in P_\kappa(\lambda)$, a bijection from $\{b_i | i < \kappa\}$ to $\{a_j | j \in X\}$. Let the black points of M_f be the set N consisting of vectors of the form $c_X + b_i + \frac{1}{2} f_X(b_i)$, and 0, and define $d_N(x)$ as the distance to N. This is a submodel of the monster.
Let $A_X = \{b_i | i < \kappa\} \cup \{a_j | j \in X\}$. Then we have that if $X \neq Y$ the types $\mathrm{tp}(c_X / A_X)$ and $\mathrm{tp}(c_Y / A_Y)$ are incompatible :

The failure of simplicity (III)

Suppose there is some c such that $\mathrm{tp}(c/A_X) = \mathrm{tp}(c_X/A_X)$ and $\mathrm{tp}(c/A_Y) = \mathrm{tp}(c_Y/A_Y)$. Take (wlog) $j \in Y \setminus X$. Pick $\ell < \kappa$ such that $f_Y(b_\ell) = a_j$. Let $k \in X$ be such that $f_X(b_\ell) = a_k$.

In M_f, the distance to black of $c_X + b_\ell - \frac{1}{2}a_k$ is 1. Therefore the distance to black of $d = c + b_\ell - \frac{1}{2}a_k$ is also 1 (in the monster) - but $e = c + b_\ell + \frac{1}{2}a_j$ is black - as $e' = c_Y + b_\ell + \frac{1}{2}a_j$ is - and the distance from e to d is $\frac{\sqrt{2}}{2} < 1$, a contradiction.

Thanks !