How Many Sprays Cover the Plane?

Ramiro de la Vega S.

Universidad de los Andes, Bogotá

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- **Definition:** $A \subseteq \mathbb{R}^2$ is a **spray** if for some $c \in \mathbb{R}^2$, $|C \cap A| < \aleph_0$ for every circle $C \subseteq \mathbb{R}^n$ centered at c.

Trivial: the plane cannot be covered by 1 spray.

Easy: the plane cannot be covered by 2 sprays.

How about 3 sprays?

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Related Results

- **Definition:** $A \subseteq \mathbb{R}^n$ is a **fog** if for some $v \in \mathbb{R}^n$, $|l \cap A| < \aleph_0$ for every line $l \subseteq \mathbb{R}^n$ parallel to v.

  Definition: $A \subseteq \mathbb{R}^n$ is a **cloud** if for some $c \in \mathbb{R}^n$, $|l \cap A| < \aleph_0$ for every line $l \subseteq \mathbb{R}^n$ with $c \in v$.

  Theorem: The following are equivalent:

  $2^{\aleph_0} = \aleph_1$.

  $\mathbb{R}^3 = A_0 \cup A_1 \cup A_2$, with A_i a fog along e_i.
  (Sierpinski, 1952)

  $\mathbb{R}^2 = A_0 \cup A_1 \cup A_2$, where each A_i is a fog.
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A General Context

- Let E_0, E_1, E_2 be three equivalence relations on $\mathbb{R}^2$ such that $|[x]_i \cap [x]_j| < \aleph_0$ for any $x \in \mathbb{R}^2$ and $i \neq j$.

  Let us say that $A \subseteq \mathbb{R}^2$ is E_i-**small** if $|[x]_i \cap A| < \aleph_0$ for every $x \in \mathbb{R}^2$.

  We want to study the statement $P(E_0, E_1, E_2)$:

  $$\exists A_0, A_1, A_2,\ \mathbb{R}^2 = \bigcup_{i \in 3} A_i \text{ and each } A_i \text{ is } E_i\text{-small.}$$

  Theorem (Erdös, Jackson, Mauldin, 1994): The following are equivalent:

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Main Result

- **Definition:** A triple $\mathbf{E} = \langle E_0, E_1, E_2 \rangle$ is **twisted** if $\forall M, N \prec V$ with $\mathbf{E} \in M \cap N$ and $N \in M$, the set

  $$\{x \in [a]_k : [x]_i \in M \setminus N,\ [x]_j \in N \setminus M\}$$

  is finite, whenever $a \in \mathbb{R}^2$ and $\{i, j, k\} = 3$.

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The Role of CH

$$\left\{x \in [a]_k : [x]_i \in M \setminus N,\ [x]_j \in N \setminus M\right\}$$

- Under CH this set is always empty:
  - If $N \cap \mathbb{R}^2$ is countable then $N \cap \mathbb{R}^2 \subseteq M$.
  - If $N \cap \mathbb{R}^2$ is uncountable then $\mathbb{R}^2 \subseteq N$.

Theorem: Under CH, every $\mathbb{E}$ is twisted.

Under $\neg CH$ a strategy to prove that certain $\mathbb{E}$ is not twisted (e.g. for Sierpinski, Davies and Komjáth´s results) is the following:

Fix $M, N \prec V$ with $|M| = \aleph_1$ and $|N| = \aleph_0$.

Find $x \in \mathbb{R}^2$ with $[x]_i \in M \setminus N$ and $[x]_j \in N \setminus M$.

Try to move x in such a way that $[x]_k$ remains constant while $[x]_i$ and $[x]_j$ change in a definable way.

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Back to Sprays

- **Theorem:** Let c_0, c_1 and c_2 be three distinct points on $\mathbb{R}^2$ lying on the same line. The following are equivalent:

  i) $2^{\aleph_0} = \aleph_1$.

  ii) $\mathbb{R}^3 = A_0 \cup A_1 \cup A_2$, A_i is a spray centered at c_i.

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- **Theorem:** The plane is the union of three sprays.

$\mathbb{R}^2$ is the union of 3 sprays (sketch)

- Let $c_0 = (-1, 0), c_1 = (1, 0)$ and $c_3 = (0, \sqrt{3})$.

  Define $(x, y) \in E_i \Leftrightarrow \|x - c_i\| = \|y - c_i\|$.

  We will identify $[x]_i$ with $\|x - c_i\|^2$.

  There is a polynomial $p \in \mathbb{R}[X, Y, Z, W]$ such that $\forall x, y \in \mathbb{R}^2$, if $[x]_2 = [y]_2$ then

  $$p([x]_0, [y]_0, [x]_1, [y]_1) = 0.$$

  If $A \subset \mathbb{R}$ is infinite then $\exists a, b \in A$ such that $p(X, Y, a, b)$ is irreducible (in $\mathbb{C}[X, Y]$).

  If $(a, b) \neq (a', b')$ and $p(X, Y, a, b)$, $p(X, Y, a', b')$ are both irreducible then the system:

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- If this set is infinite, fix x, y in it such that
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By elementarity there exist $x', y' \in M$ such that $[x']_1, [y']_1 \in N$, $[x']_0 = [x]_0$, $[y']_0 = [y]_0$ and
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Note that $([x']_1, [y']_1) \neq ([x]_1, [y]_1)$ since the first one is in M and the second one is not.

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