How Many Sprays Cover the Plane?

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  Trivial: the plane cannot be covered by 1 spray.

  Easy: the plane cannot be covered by 2 sprays.

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Related Results

• **Definition:** $A \subseteq \mathbb{R}^n$ is a fog if for some $v \in \mathbb{R}^n$, $|l \cap A| < \aleph_0$ for every line $l \subseteq \mathbb{R}^n$ parallel to $v$.

**Definition:** $A \subseteq \mathbb{R}^n$ is a cloud if for some $c \in \mathbb{R}^n$, $|l \cap A| < \aleph_0$ for every line $l \subseteq \mathbb{R}^n$ with $c \in v$.

**Theorem:** The following are equivalent:

$2^{\aleph_0} = \aleph_1$.

$\mathbb{R}^3 = A_0 \cup A_1 \cup A_2$, with $A_i$ a fog along $e_i$.  
(Sierpinski, 1952)

$\mathbb{R}^2 = A_0 \cup A_1 \cup A_2$, with each $A_i$ is a fog.  
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A General Context

- Let $E_0, E_1, E_2$ be three equivalence relations on $\mathbb{R}^2$ such that $|[x]_i \cap [x]_j| < \aleph_0$ for any $x \in \mathbb{R}^2$ and $i \neq j$.

Let us say that $A \subseteq \mathbb{R}^2$ is $E_i$-small if $|[x]_i \cap A| < \aleph_0$ for every $x \in \mathbb{R}^2$.

We want to study the statement $P(E_0, E_1, E_2)$:

$\exists A_0, A_1, A_2, \mathbb{R}^2 = \bigcup_{i \in 3} A_i$ and each $A_i$ is $E_i$-small.

Theorem (Erdös, Jackson, Mauldin, 1994): The following are equivalent:

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$P(E_0, E_1, E_2)$ holds for every $E_0, E_1, E_2$. 
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i) $2^{\aleph_0} = \aleph_1$.

ii) $P(E_0, E_1, E_2)$ holds for every $E_0, E_1, E_2$. 
Main Result

- **Definition**: A triple $E = \langle E_0, E_1, E_2 \rangle$ is twisted if $\forall M, N \prec V$ with $E \in M \cap N$ and $N \in M$, the set
  \[
  \{ x \in [a]_k : [x]_i \in M \setminus N, \ [x]_j \in N \setminus M \}
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  is finite, whenever $a \in \mathbb{R}^2$ and $\{i, j, k\} = 3$.

**Theorem**: $P(E) \iff E$ is twisted.
Main Result

- **Definition:** A triple $E = \langle E_0, E_1, E_2 \rangle$ is **twisted** if $\forall M, N \prec V$ with $E \in M \cap N$ and $N \in M$, the set
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- **Theorem:** $P(E) \iff E$ is twisted.
The Role of \( CH \)

\[ \{ x \in [a]_k : [x]_i \in M \setminus N, \ [x]_j \in N \setminus M \} \]

- Under \( CH \) this set is always empty:
  - If \( N \cap \mathbb{R}^2 \) is countable then \( N \cap \mathbb{R}^2 \subseteq M \).
  - If \( N \cap \mathbb{R}^2 \) is uncountable then \( \mathbb{R}^2 \subseteq N \).

**Theorem:** Under \( CH \), every \( E \) is twisted.

Under \( \neg CH \) a strategy to prove that certain \( E \) is not twisted (e.g. for Sierpinski, Davies and Komjáth’s results) is the following:

Fix \( M, N \prec V \) with \( |M| = \aleph_1 \) and \( |N| = \aleph_0 \).

Find \( x \in \mathbb{R}^2 \) with \( [x]_i \in M \setminus N \) and \( [x]_j \in N \setminus M \).

Try to move \( x \) in such a way that \( [x]_k \) remains constant while \( [x]_i \) and \( [x]_j \) change in a definable way.
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Back to Sprays

- **Theorem:** Let \( c_0, c_1 \) and \( c_2 \) be three distinct points on \( \mathbb{R}^2 \) lying on the same line. The following are equivalent:

  i) \( 2^{\aleph_0} = \aleph_1 \).

  ii) \( \mathbb{R}^3 = A_0 \cup A_1 \cup A_2, \ A_i \) is a spray centered at \( c_i \).

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• **Theorem:** The plane is the union of three sprays.
$\mathbb{R}^2$ is the union of 3 sprays (sketch)

- Let $c_0 = (-1, 0), c_1 = (1, 0)$ and $c_3 = (0, \sqrt{3})$.

Define $(x, y) \in E_i \iff \|x - c_i\| = \|y - c_i\|$.

We will identify $[x]_i$ with $\|x - c_i\|^2$.

There is a polynomial $p \in \mathbb{R}[X, Y, Z, W]$ such that
\[ \forall x, y \in \mathbb{R}^2, \text{ if } [x]_2 = [y]_2 \text{ then } p([x]_0, [y]_0, [x]_1, [y]_1) = 0. \]

If $A \subset \mathbb{R}$ is infinite then $\exists a, b \in A$ such that $p(X, Y, a, b)$

is irreducible (in $\mathbb{C}[X, Y]$).

If $(a, b) \neq (a', b')$ and $p(X, Y, a, b), p(X, Y, a', b')$ are both irreducible then the system:
\[ p(X, Y, a, b) = 0 \]
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has only finitely many solutions.
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- If this set is infinite, fix $x, y$ in it such that 
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By elementarity there exist $x', y' \in M$ such that 
\[ [x']_1, [y']_1 \in N, \ [x']_0 = [x]_0, \ [y']_0 = [y]_0 \] and 
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Note that $([x']_1, [y']_1) \neq ([x]_1, [y]_1)$ since the first one is in $M$ and the second one is not.

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is definable in $N$, $([x]_0, [y]_0)$ is one of its finitely many solutions, but $([x]_0, [y]_0) \notin N$. 
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$$\{x \in [a]_2 : [x]_0 \in M \setminus N, \ [x]_1 \in N \setminus M\}$$

- If this set is infinite, fix $x, y$ in it such that $p(X, Y, [x]_1, [y]_1)$ is irreducible.

- By elementarity there exist $x', y' \in M$ such that $[x']_1, [y']_1 \in N,$ $[x']_0 = [x]_0,$ $[y']_0 = [y]_0$ and $p(X, Y, [x']_1, [y']_1)$ is irreducible.

- Note that $([x']_1, [y']_1) \neq ([x]_1, [y]_1)$ since the first one is in $M$ and the second one is not.

- The system
  $$p(X, Y, [x]_1, [y]_1) = 0$$
  $$p(X, Y, [x']_1, [y']_1) = 0$$

  is definable in $N,$ $([x]_0, [y]_0)$ is one of its finitely many solutions, but $([x]_0, [y]_0) \notin N.$
THE END