

Locally Connected HS/HL Compacta

Question [M.E. Rudin, 1982]: Is there a compact X which is

- (1) non-metrizable, and
- (2) locally connected, and
- (3) hereditarily Lindelöf (HL)?
- (4) and hereditarily separable (HS) ?

Answer Φ ullunnova: (CH) Yes.

But under PFA his construction crumbles.

Kunen's Answer: $\text{Con}(\text{MA} + \neg\text{CH} + \text{Yes})$.

ZFC Answer: ???

Maybe: look at maps $f : X \rightarrow Y$, for Y compact metric.

For $X \in M \prec H(\theta)$, let $\pi : X \rightarrow X/M$ denote the quotient map given by $\pi(x) = \pi(y) \iff f(x) = f(y) \quad \forall f \in M \cap C(X, [0, 1])$.

Lemma. Suppose X is compact, $w(X) = \aleph_1$, and $\chi(X) = \aleph_0$. Then X is an *Aronszajn compactum* iff whenever M is countable, $X \in M \prec H(\theta)$, and $\pi : X \rightarrow X/M$ is the usual quotient map, $\pi^{-1}\{y\}$ is a singleton for all but countably many $y \in X/M$.

Theorem (\diamond) There is an Aronszajn compactum X which is HS and HL. X can be constructed to be locally connected and connected, or to be totally disconnected.

Question: Is there (in ZFC) an HL Aronszajn compactum?

Reference: J. Hart and K. Kunen, Aronszajn Compacta
http://www.uwosh.edu/faculty_staff/hartj/

Outline of this talk:

1. Background: The properties in the question
2. Aronszajn compacta
 - a. The definition
 - b. Elementary examples and non-examples
 - c. Our \diamond Aronsajn spaces
 - d. Sketch of \diamond construction of Aronszajn spaces
 - e. Recap of open questions

All spaces are Hausdorff.

The properties in the question:

Rudin's Question+: Is there a compact X which is

- (1) non-metrizable, and
- (2) locally connected, and
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(2) locally connected:

Juhász' Question: If X is locally connected and compact and Y is T_2 , is every preserving function $f : X \rightarrow Y$ continuous?

Def. A function $f : X \rightarrow Y$ is *preserving* whenever the image of every compact subspace is compact and the image of every connected subspace is connected.

Reference: J. Gerlits, I. Juhász, L. Soukup, and Z. Szentmiklóssy, Characterizing continuity by preserving compactness and connectedness, *Topology Appl.* 138 (2004), no. 1-3, 21–44.

Remarks:

A continuous function is always preserving.

Locally connected is necessary:

White [1971]: For Tychonov spaces X and Y , if X is not locally connected at a point p in X , then there is a preserving function $f : X \rightarrow Y$ that is not continuous at p .

but not sufficient:

McMillan [1970] constructed a locally connected hedgehog space X , with a preserving function $f : X \rightarrow [0, 1]$ that is not continuous.

(1) non-metrizable

Whyburn [1965] If X is locally connected and first countable and Y is T_2 , then every preserving function $f : X \rightarrow Y$ is continuous.

The properties (continued):

(1) non-metrizable (continued)

Juhász' other Question: Is there a locally connected continuum without nontrivial convergent sequences?

Def. A *continuum* is a compact connected space.

van Mill's Answer: (CH) Yes.

There is an eXample with $\dim(X) = \infty$.

Reference: J. van Mill, A locally connected continuum without convergent sequences, *Topology Appl.* 126 (2002), no. 1-2, 273-280.

Our Answer: (\diamond) Yes.

Reference: J. Hart and K. Kunen, One dimensional locally connected S-spaces, *Topology Appl.* 156 (2009) 601-609.

(4) Hereditarily Separable (HS):

Our \diamond example is also HS.

But any compactum having no convergent ω -sequences has points of uncountable character, and hence is not HL.

But that doesn't rule out HL for Juhász' preserving question:

(3) Hereditarily Lindelöf (HL):

Gerlits, J., Juhász, I., Soukup, L., and Szentmiklóssy, Z., include an HL eXample with a discontinuous preserving function $f : X \rightarrow [0, 1]$.

More reasons to consider **(3)**, **(4)**: they keep appearing in the lit:

surveys:	papers:
Juhász [1978]	Hajnal and Juhasz [1968,1973,1974]
M.E.Rudin [1980]	M.E.Rudin [1972]
Roitman [1984]	Fedorchuk [1975]
Todorčević [1984]	Kunen [1977,1981]
Todorčević [1989]	Džamonja and Kunen [1993]
	Moore [2006]

The definition of Aronszajn compact:

First, a little notation: For $X \subseteq [0, 1]^{\omega_1}$ and $\alpha \leq \beta \leq \omega_1$:

$\pi_\alpha^\beta : [0, 1]^\beta \rightarrow [0, 1]^\alpha$ is the natural projection,

$X_\alpha = \pi_\alpha^{\omega_1}(X)$, and $\sigma_\alpha^\beta = \pi_\alpha^\beta \upharpoonright X_\beta$.

An *embedded Aronszajn compactum* is a closed $X \subseteq [0, 1]^{\omega_1}$ with $w(X) = \aleph_1$ and $\chi(X) = \aleph_0$ such that for some club $C \subseteq \omega_1$:

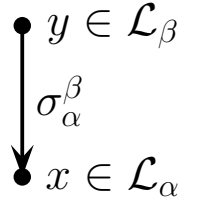
for each $\alpha \in C$ $\mathcal{L}_\alpha := \{x \in X_\alpha : |(\sigma_\alpha^{\omega_1})^{-1}\{x\}| > 1\}$ is countable.

For each such X , define $T = T(X) := \bigcup \{\mathcal{L}_\alpha : \alpha \in C\}$,

and let \triangleleft denote the following order:

if $\alpha, \beta \in C$, $\alpha < \beta$, $x \in \mathcal{L}_\alpha$ and $y \in \mathcal{L}_\beta$,

then $x \triangleleft y$ iff $x = \sigma_\alpha^\beta(y)$.



Remark: $\langle T(X), \triangleleft \rangle$ is an Aronszajn tree.

Each level \mathcal{L}_α is countable by definition,

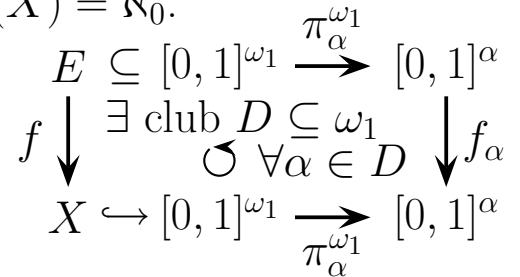
each $\mathcal{L}_\alpha \neq \emptyset$ because $w(X) = \aleph_1$,

and every chain in T is countable because $\chi(X) = \aleph_0$.

Def. An *Aronszajn compactum*

is a compact space X homeomorphic

to an embedded Aronszajn compactum E .



An Aronszajn line may or may not yield an Aronszajn compactum:

An *Aronszajn line* is a LOTS of size \aleph_1 , with no increasing or

decreasing ω_1 -sequences, and no uncountable subsets of real type.

Of *real type* means order-isomorphic to a subset of \mathbb{R} .

Def. A *compact* Aronszajn line is a compact LOTS X such that

$w(X) = \aleph_1$ and $\chi(X) = \aleph_0$ and the closure of every countable set is

second countable.

Lemma A LOTS X is an Aronszajn compactum

iff X is a *compact* Aronszajn line.

Examples and non-examples of Aronszajn compacta:

Example:

☞ The Dedekind completion X of an Aronszajn line:
its tree $T(X)$ is essentially the standard tree of closed intervals.

A *Suslin line* is any LOTS which is ccc and not separable.

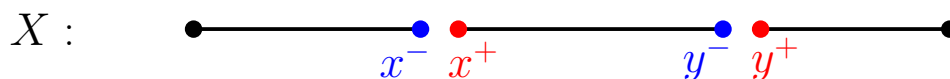
If it's compact, it may or may not be an Aronszajn compactum:

Lemma Let X be a compact Suslin line. Then X is a compacted Aronszajn line iff $D := \{x \in X : \exists y > x ([x, y] = \{x, y\})\}$ does not contain an uncountable subset of real type.

Non-example:

⚡ Form X from a connected compact Suslin line Y by doubling uncountably many points lying in some Cantor subset C of Y :

Mimic the double arrow space [Alexandroff and Urysohn, 1929], but start with Y , replace each $x \in C$ by a pair $x^- < x^+$, and refine the order topology of Y .



Theorem (\diamond) For each of the following $2 \cdot 3 = 6$ possibilities, there is an HS, HL, Aronszajn compactum X with tree $T = T(X)$.

Possibilities for T :

- a. T is Suslin.
- b. T is special (T is the union of ω antichains).

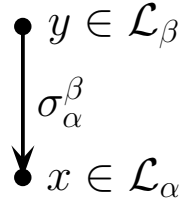
Possibilities for X :

- α . $\dim(X) = 0$.
- β . $\dim(X) = 1$ and X is connected and locally connected.
- γ . $\dim(X) = \infty$ and X is connected and locally connected.

Sketch of an Aronszajn compactum X that is locally connected, HL, HS, and with $\dim(X) = 1$:

$X \subseteq [0, 1]^{\omega_1}$ is an Aronszajn compactum iff $w(X) = \aleph_1$, $\chi(X) = \aleph_0$, and for some club $C \subseteq \omega_1$: $\mathcal{L}_\alpha := \{x \in X_\alpha : |(\sigma_\alpha^{\omega_1})^{-1}\{x\}| > 1\}$ is countable for each $\alpha \in C$.

$T(X) := \bigcup\{\mathcal{L}_\alpha : \alpha \in C\}$, and for $x, y \in T(X)$:
 $x \triangleleft y$ iff $x \in \mathcal{L}_\alpha$, $y \in \mathcal{L}_\beta$, $\alpha < \beta$, and $x = \sigma_\alpha^\beta(y)$.



To simplify notation, let $Q = [0, 1]^\omega$.

Obtain $X = X_{\omega_1} \subset Q^{\omega_1}$ by an inductive construction, and form $T(X)$ by an inductive Aronszajn tree construction.

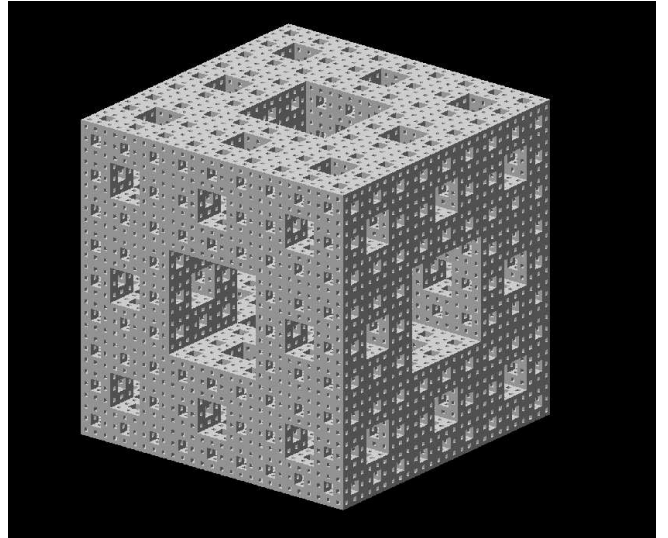
At stage $\alpha < \omega_1$: Determine the projection X_α of X on Q^α .

Select a countable set $\mathcal{L}_\alpha \subseteq X_\alpha$ of “expandable points”. So for $\beta > \alpha$, whenever $x \notin \mathcal{L}_\alpha$, construct $X_\beta \subseteq Q^\beta$, so that $|(\sigma_\alpha^\beta)^{-1}\{x\}| = 1$.

Then, X is the inverse limit of $\langle X_\alpha : \alpha < \omega_1 \rangle$.

To make $\dim(X) = 1$: set $X_1 = \text{MS}$.

The Menger Sponge (MS) is, up to homeomorphism, the only one-dimensional Peano continuum (connected, locally connected, compact metric space) with no locally separating points and no non-empty planar open sets.



Pix thanx to <http://www.joachim-reichel.de/>

Sketch (continued):

To start:

Let \mathcal{L}_1 be any countable dense subset of X_1 .

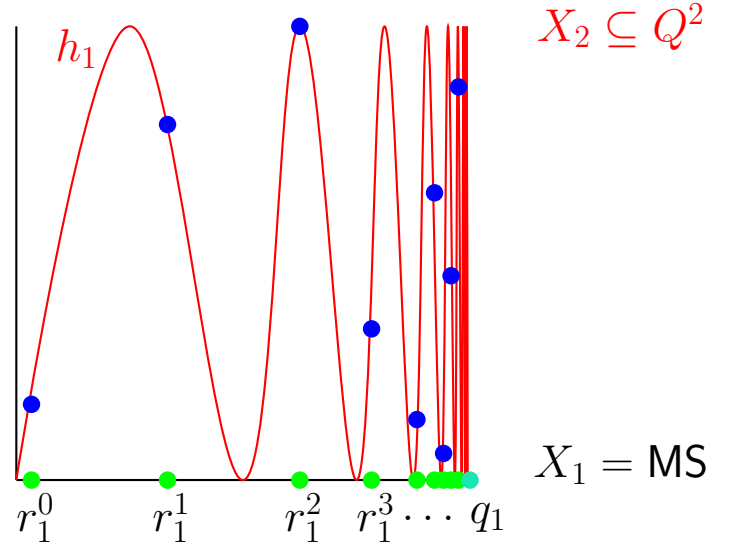
To get $X_2 \subseteq Q^2$:

Choose h_1, q_1 , and r_1^n , for $n < \omega$, so that:

(a) $q_1 \in \mathcal{L}_1$,
 $h_1 \in C(X_1 \setminus \{q_1\}, [0, 1])$.

(b) $r_1^n \in X_1 \setminus \{q_1\}$,
 $\langle r_1^n : n \in \omega \rangle \rightarrow q_1$, and
 $[0, 1] = \overline{\{h_1(r_1^n) : n \in \omega\}}$.

Let $X_2 = \overline{h_1} \subseteq Q^2$.



Let \mathcal{D}_2 be any subset of $(\sigma_1^2)^{-1}\{q_1\}$ with $2 \leq |\mathcal{D}_2| \leq \aleph_0$.

Let $\mathcal{L}_2 = (\sigma_1^2)^{-1}(\mathcal{L}_1 \setminus \{q_1\}) \cup \mathcal{D}_2$.

Observe:

X_2 is connected and has no isolated points.

$\{q_1\} \times [0, 1]$ is closed and connected in X_2 .

\mathcal{L}_2 is a countable dense subset of X_2 .

The projection $\sigma_1^2 : X_2 \twoheadrightarrow X_1$ is irreducible.

Def. A map $f : X \twoheadrightarrow Y$ is *irreducible* iff $\forall A \overset{\text{closed}}{\subsetneq} X \quad f(A) \neq Y$.

Sketch (continued):

To cultivate the tree $T(X)$:

At stage $\beta = \alpha + 1 \geq 2$, q_α branches into at least 2 new points:

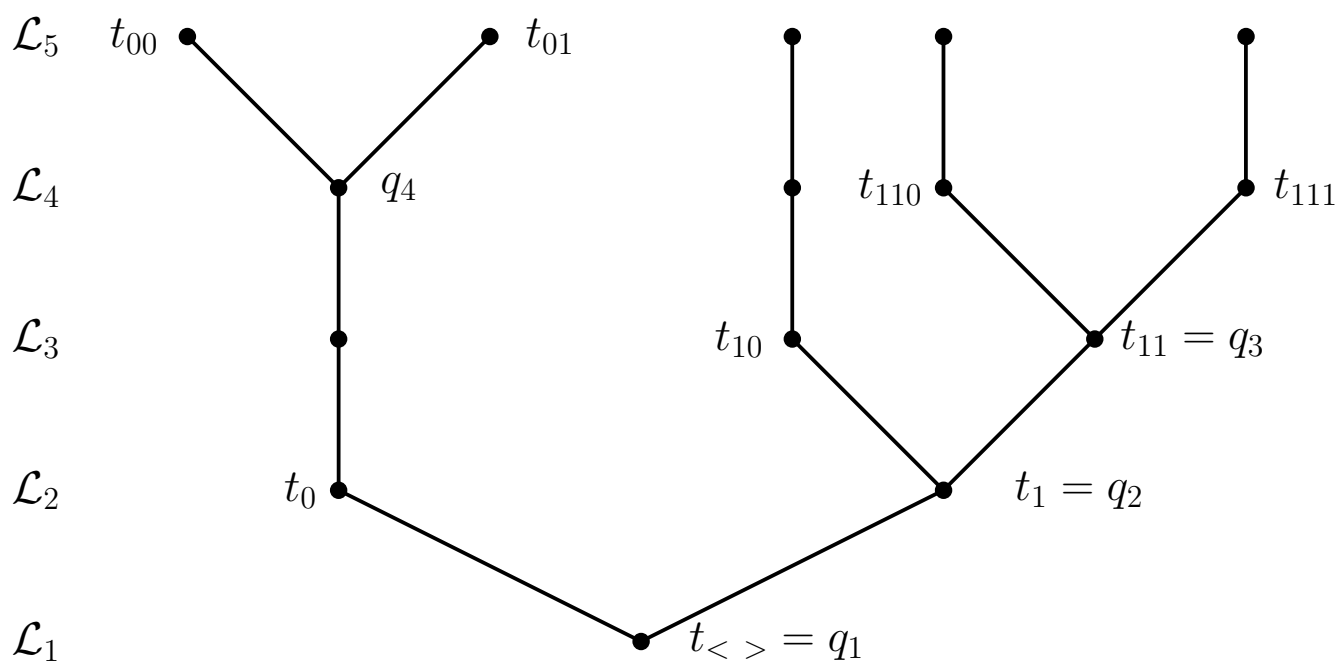
\mathcal{D}_β is any subset of $(\sigma_\alpha^\beta)^{-1}\{q_\alpha\}$ such that $2 \leq |\mathcal{D}_\beta| \leq \aleph_0$,

and $\mathcal{L}_\beta = (\sigma_\alpha^\beta)^{-1}(\mathcal{L}_\alpha \setminus \{q_\alpha\}) \cup \mathcal{D}_\beta$.

Moreover, we leaf \mathfrak{c} choices for limit stage expandable points:

Every $x \in \mathcal{L}_\alpha$ gets expanded by stage $\alpha + n$ for some $n \in \omega$:

put $q_{\alpha+n} \in \mathcal{L}_{\alpha+n}$ so that $\sigma_\alpha^{\alpha+n}(q_{\alpha+n}) = x$.



To make X HL:

Use \diamond to choose a closed $\tilde{F}_\alpha \subseteq Q^\alpha$ for each $\alpha < \omega_1$, so that $\{\alpha < \omega_1 : \pi_\alpha^{\omega_1}(F) = \tilde{F}_\alpha\}$ is stationary for all closed $F \subseteq Q^{\omega_1}$.
 Let $F_\beta = \tilde{F}_\beta$ if $\tilde{F}_\beta \subseteq X_\beta$ and β is a limit; otherwise, let $F_\beta = \emptyset$.
 To ensure all closed sets will be G_δ 's, keep the F_β nice:
 let $\mathcal{P}_\beta = \{F_\beta\} \cup \{(\sigma_\alpha^\beta)^{-1}(P) : 0 < \alpha < \beta \text{ \& } P \in \mathcal{P}_\alpha\}$.
 and make sure $\sigma_\alpha^\beta \upharpoonright ((\sigma_\alpha^\beta)^{-1}(P)) : (\sigma_\alpha^\beta)^{-1}(P) \rightarrow P$ is irreducible
 whenever $\alpha \leq \beta \leq \omega_1$ and $P \in \mathcal{P}_\alpha$.

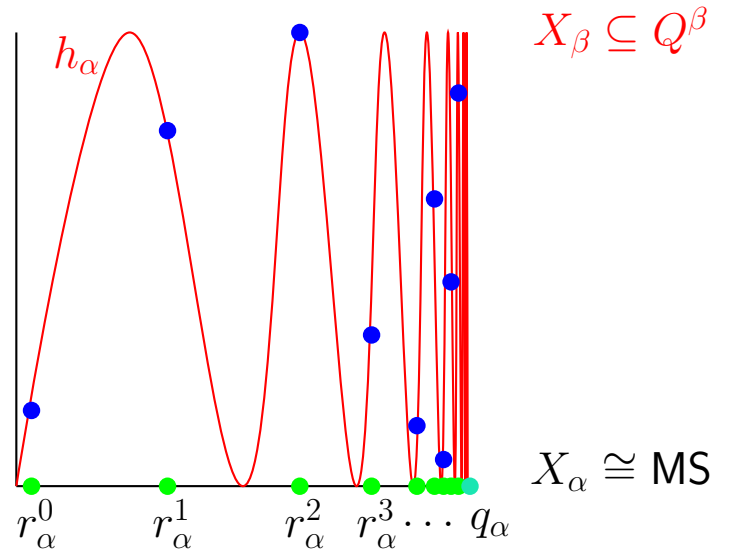
Each \mathcal{P}_α is countable, so we can select the set of expandable points $\mathcal{L}_\alpha \subseteq X_\alpha$ so that $\mathcal{L}_\alpha \cap (P \setminus \ker(P)) = \emptyset$ whenever $P \in \mathcal{P}_\alpha$.

At stage $\beta = \alpha + 1$, proceed as for $\beta = 2$,

and, if $q_\alpha \in P \in \mathcal{P}_\alpha$,

choose $r_\alpha^n \in \ker(P)$

for infinitely many n .



For limit β , let $\mathcal{L}_\beta = \{x^* : x \in \bigcup_{\alpha < \beta} \mathcal{L}_\alpha\}$, where, x^* , for $x \in \mathcal{L}_\alpha$, is some $y \in X_\beta$ such that $\sigma_\alpha^\beta(y) = x$ and $\sigma_\xi^\beta(y) \in \mathcal{L}_\xi$ for all $\xi < \beta$.

Recap of Open questions:

Rudin's Question+: Is there a compact X which is

- (1) non-metrizable, and
- (2) locally connected, and
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Question: Is there, in ZFC, an HL Aronszajn compactum?

To refute the existence of an HL Aronszajn compactum, one needs more than just an Aronszajn tree of closed sets, since this much exists in the Cantor set:

Proposition There is an Aronszajn tree T whose nodes are closed subsets of the Cantor set 2^ω . The tree ordering is \supset , with root 2^ω . Each level of T consists of a pairwise disjoint family of sets.

Proof: it's like that of Theorem 4 of Galvin and Miller which is attributed there to Todorčević.

Reference: F. Galvin and A. Miller, γ -sets and other singular sets of real numbers, *Topology Appl.* 17 (1984) 145-155.