Locally Connected HS/HL Compacta

**Question** [M.E. Rudin, 1982]: Is there a compact $X$ which is
1) non-metrizable, and
2) locally connected, and
3) hereditarily Lindelöf (HL)?
4) and hereditarily separable (HS)?

**Answer** Полицена: (CH) Yes.
But under PFA his construction crumbles.

**Kunen’s Answer:** Con(MA + ¬CH + Yes).

**ZFC Answer:** ???
Maybe: look at maps $f : X \to Y$, for $Y$ compact metric.
For $X \in M \prec H(\theta)$, let $\pi : X \to X/M$ denote the quotient map
given by $\pi(x) = \pi(y) \iff f(x) = f(y) \forall f \in M \cap C(X, [0, 1])$.

**Lemma.** Suppose $X$ is compact, $w(X) = \aleph_1$, and $\chi(X) = \aleph_0$.
Then $X$ is an *Aronszajn compactum* iff whenever $M$ is countable,
$X \in M \prec H(\theta)$, and $\pi : X \to X/M$ is the usual quotient map,
$\pi^{-1}\{y\}$ is a singleton for all but countably many $y \in X/M$.

**Theorem** (♦) There is an Aronszajn compactum $X$ which is HS and HL. $X$ can be constructed to be locally connected and connected, or to be totally disconnected.

**Question:** Is there (in ZFC) an HL Aronszajn compactum?

**Reference:** J. Hart and K. Kunen, Aronszajn Compacta
http://www.uwosh.edu/faculty_staff/hartj/
Outline of this talk:

1. Background: The properties in the question
2. Aronszajn compacta
   a. The definition
   b. Elementary examples and non-examples
   c. Our $\Diamond$ Aronsajn spaces
   d. Sketch of $\Diamond$ construction of Aronszajn spaces
   e. Recap of open questions

All spaces are Hausdorff.
The properties in the question:

**Rudin’s Question**: Is there a compact $X$ which is
(1) non-metrizable, and
(2) locally connected, and
(3) hereditarily Lindelöf (HL)?
(4) and hereditarily separable (HS)?

**Locally connected**:

**Juhász’ Question**: If $X$ is locally connected and compact and $Y$ is $T_2$, is every preserving function $f : X \rightarrow Y$ continuous?

*Def.* A function $f : X \rightarrow Y$ is *preserving* whenever the image of every compact subspace is compact and the image of every connected subspace is connected.


*Remarks*:
A continuous function is always preserving.

Locally connected is necessary:

White [1971]: For Tychonov spaces $X$ and $Y$, if $X$ is not locally connected at a point $p$ in $X$, then there is a preserving function $f : X \rightarrow Y$ that is not continuous at $p$.

but not sufficient:

McMillan [1970] constructed a locally connected hedgehog space $X$, with a preserving function $f : X \rightarrow [0, 1]$ that is not continuous.

**Non-metrizable**

Whyburn [1965] If $X$ is locally connected and first countable and $Y$ is $T_2$, then every preserving function $f : X \rightarrow Y$ is continuous.
The properties (continued):

(1) non-metrizable (continued)

Juhász’ other Question: Is there a locally connected continuum without nontrivial convergent sequences?

Def. A continuum is a compact connected space.

van Mill’s Answer: (CH) Yes.

There is an example with \( \dim(X) = \infty \).


Our Answer: (◊) Yes.


(4) Hereditarily Separable (HS):

Our ◊ example is also HS.

But any compactum having no convergent \( \omega \)-sequences has points of uncountable character, and hence is not HL.

But that doesn’t rule out HL for Juhász’ preserving question:

(3) Hereditarily Lindelöf (HL):

Gerlits, J., Juhász, I., Soukup, L., and Szentmiklóssy, Z., include an HL example with a discontinuous preserving function \( f : X \to [0, 1] \).

More reasons to consider (3), (4): they keep appearing in the lit:

surveys:

Juhász [1978]
M.E.Rudin [1980]
Roitman [1984]
Todorčević [1984]
Todorčević [1989]

papers:

M.E.Rudin [1972]
Fedorchuk [1975]
Kunen [1977,1981]
Džamonja and Kunen [1993]
Moore [2006]
The definition of Aronszajn compact:

First, a little notation: For $X \subseteq [0, 1]^{\omega_1}$ and $\alpha \leq \beta \leq \omega_1$: 

- $\pi_\alpha^\beta : [0, 1]^\beta \rightarrow [0, 1]^\alpha$ is the natural projection,
- $X_\alpha = \pi_\alpha^\omega_1(X)$, and $\sigma_\alpha^\beta = \pi_\alpha^\beta|X_\beta$.

An embedded Aronszajn compactum is a closed $X \subseteq [0, 1]^{\omega_1}$ with $w(X) = \aleph_1$ and $\chi(X) = \aleph_0$ such that for some club $C \subseteq \omega_1$:

For each $\alpha \in C$, $L_\alpha := \{ x \in X_\alpha : |(\sigma_\alpha^{\omega_1})^{-1}\{x\}| > 1 \}$ is countable.

For each such $X$, define $T = T(X) := \bigcup \{ L_\alpha : \alpha \in C \}$, and let $\triangleleft$ denote the following order:

- if $\alpha, \beta \in C$, $\alpha < \beta$, $x \in L_\alpha$ and $y \in L_\beta$,
- then $x \triangleleft y$ iff $x = \sigma_\alpha^\beta(y)$.

Remark: $\langle T(X), \triangleleft \rangle$ is an Aronszajn tree.

Each level $L_\alpha$ is countable by definition, each $L_\alpha \neq \emptyset$ because $w(X) = \aleph_1$,

and every chain in $T$ is countable because $\chi(X) = \aleph_0$.

Def. An Aronszajn compactum is a compact space $X$ homeomorphic to an embedded Aronszajn compactum $E$.

An Aronszajn line may or may not yield an Aronszajn compactum:

An Aronszajn line is a LOTS of size $\aleph_1$, with no increasing or decreasing $\omega_1$-sequences, and no uncountable subsets of real type.

Of real type means order-isomorphic to a subset of $\mathbb{R}$.

Def. A compacted Aronszajn line is a compact LOTS $X$ such that $w(X) = \aleph_1$ and $\chi(X) = \aleph_0$ and the closure of every countable set is second countable.

Lemma A LOTS $X$ is an Aronszajn compactum iff $X$ is a compacted Aronszajn line.
Examples and non-examples of Aronszajn compacta:

Example:
☞ The Dedekind completion $X$ of an Aronszajn line:
its tree $T(X)$ is essentially the standard tree of closed intervals.

A Suslin line is any LOTS which is ccc and not separable.
If it’s compact, it may or may not be an Aronszajn compactum:

Lemma Let $X$ be a compact Suslin line. Then $X$ is a compacted
Aronszajn line iff $D := \{x \in X : \exists y > x ([x, y] = \{x, y\})\}$ does not
contain an uncountable subset of real type.

Non-example:
☞ Form $X$ from a connected compact Suslin line $Y$ by doubling
uncountably many points lying in some Cantor subset $C$ of $Y$:
Mimic the double arrow space [Alexandroff and Urysohn, 1929],
but start with $Y$, replace each $x \in C$ by a pair $x^- < x^+$,
and refine the order topology of $Y$.

Theorem (◊) For each of the following $2 \cdot 3 = 6$ possibilities, there
is an HS, HL, Aronszajn compactum $X$ with tree $T = T(X)$.
Possibilities for $T$:
  a. $T$ is Suslin.
  b. $T$ is special ($T$ is the union of $\omega$ antichains).
Possibilities for $X$:
  a. $\dim(X) = 0$.
  $\beta$. $\dim(X) = 1$ and $X$ is connected and locally connected.
  $\gamma$. $\dim(X) = \infty$ and $X$ is connected and locally connected.
Sketch of an Aronszajn compactum $X$ that is locally connected, HL, HS, and with $\dim(X) = 1$:

$X \subseteq [0,1]^{\omega_1}$ is an Aronszajn compactum iff $w(X) = \aleph_1$, $\chi(X) = \aleph_0$, and for some club $C \subseteq \omega_1$: $\mathcal{L}_\alpha := \{x \in X_\alpha : |(\sigma_\alpha^{\omega_1})^{-1}\{x\}| > 1\}$ is countable for each $\alpha \in C$.

$T(X) := \bigcup\{\mathcal{L}_\alpha : \alpha \in C\}$, and for $x, y \in T(X)$:

$x \triangleleft y$ iff $x \in \mathcal{L}_\alpha$, $y \in \mathcal{L}_\beta$, $\alpha < \beta$, and $x = \sigma_\alpha^\beta(y)$.

To simplify notation, let $Q = [0,1]^{\omega}$. Obtain $X = X_{\omega_1} \subset Q^{\omega_1}$ by an inductive construction, and form $T(X)$ by an inductive Aronszajn tree construction.

At stage $\alpha < \omega_1$: Determine the projection $X_\alpha$ of $X$ on $Q^\alpha$.

Select a countable set $\mathcal{L}_\alpha \subseteq X_\alpha$ of “expandable points”. So for $\beta > \alpha$, whenever $x \notin \mathcal{L}_\alpha$, construct $X_\beta \subseteq Q^\beta$, so that $|(\sigma_\alpha^\beta)^{-1}\{x\}| = 1$.

Then, $X$ is the inverse limit of $\langle X_\alpha : \alpha < \omega_1 \rangle$.

To make $\dim(X) = 1$: set $X_1 = MS$.

The Menger Sponge ($MS$) is, up to homeomorphism, the only one-dimensional Peano continuum (connected, locally connected, compact metric space) with no locally separating points and no non-empty planar open sets.

Pix thanx to http://www.joachim-reichel.de/
**Sketch (continued):**

To start:

Let $L_1$ be any countable dense subset of $X_1$.

To get $X_2 \subseteq Q^2$:

Choose $h_1, q_1$, and $r^n_1$, for $n < \omega$, so that:

(a) $q_1 \in L_1$,  
   $h_1 \in C(X_1 \{q_1\}, [0, 1])$.

(b) $r^n_1 \in X_1 \{q_1\}$,  
   $\langle r^n_1 : n \in \omega \rangle \to q_1$, and  
   $[0, 1] = \{h_1(r^n_1) : n \in \omega\}$.

Let $X_2 = \overline{h_1} \subseteq Q^2$.

Let $D_2$ be any subset of $(\sigma^2_1)^{-1}\{q_1\}$ with $2 \leq |D_2| \leq \aleph_0$.

Let $L_2 = (\sigma^2_1)^{-1}(L_1 \{q_1\}) \cup D_2$.

Observe:

- $X_2$ is connected and has no isolated points.
- $\{q_1\} \times [0, 1]$ is closed and connected in $X_2$.
- $L_2$ is a countable dense subset of $X_2$.
- The projection $\sigma^2_1 : X_2 \to X_1$ is irreducible.

**Def.** A map $f : X \to Y$ is *irreducible* iff $\forall A \subseteq X \text{ closed } f(A) \neq Y$.
Sketch (continued):

To cultivate the tree $T(X)$:
At stage $\beta = \alpha + 1 \geq 2$, $q_\alpha$ branches into at least 2 new points: $D_\beta$ is any subset of $(\sigma_\alpha^\beta)^{-1}\{q_\alpha\}$ such that $2 \leq |D_\beta| \leq \aleph_0$, and $L_\beta = (\sigma_\alpha^\beta)^{-1}(L_\alpha \setminus \{q_\alpha\}) \cup D_\beta$.
Moreover, we leaf $c$ choices for limit stage expandable points:
Every $x \in L_\alpha$ gets expanded by stage $\alpha + n$ for some $n \in \omega$:
put $q_\alpha + n \in L_{\alpha + n}$ so that $\sigma_\alpha^{\alpha+n}(q_\alpha+n) = x$. 

\begin{itemize}
  \item $L_5$
  \item $t_{00}$
  \item $t_{01}$
  \item $L_4$
  \item $q_4$
  \item $t_{10}$
  \item $t_{110}$
  \item $t_{111}$
  \item $L_3$
  \item $t_0$
  \item $t_1 = q_2$
  \item $t_{11} = q_3$
  \item $L_2$
  \item $t_{< >} = q_1$
  \item $L_1$
\end{itemize}
To make $X$ HL:

Use ♦ to choose a closed $\widetilde{F}_\alpha \subseteq Q^\alpha$ for each $\alpha < \omega_1$, so that
\[ \{ \alpha < \omega_1 : \pi^\omega_1_\alpha(F) = \widetilde{F}_\alpha \} \] is stationary for all closed $F \subseteq Q^{\omega_1}$.
Let $F_\beta = \widetilde{F}_\beta$ if $\widetilde{F}_\beta \subseteq X_\beta$ and $\beta$ is a limit; otherwise, let $F_\beta = \emptyset$.

To ensure all closed sets will be $G_\delta$’s, keep the $F_\beta$ nice:
let $\mathcal{P}_\beta = \{ F_\beta \} \cup \{ (\sigma_\alpha^\beta)^{-1}(P) : 0 < \alpha < \beta \land P \in \mathcal{P}_\alpha \}$.

Each $\mathcal{P}_\alpha$ is countable, so we can select the set of expandable points $\mathcal{L}_\alpha \subseteq X_\alpha$ so that $\mathcal{L}_\alpha \cap (P \setminus \ker(P)) = \emptyset$ whenever $P \in \mathcal{P}_\alpha$.

At stage $\beta = \alpha + 1$, proceed as for $\beta = 2$,
and, if $q_\alpha \in P \in \mathcal{P}_\alpha$,
choose $r^n_\alpha \in \ker(P)$
for infinitely many $n$.

For limit $\beta$, let $\mathcal{L}_\beta = \{ x^* : x \in \bigcup_{\alpha < \beta} \mathcal{L}_\alpha \}$, where, $x^*$, for $x \in \mathcal{L}_\alpha$, is some $y \in X_\beta$ such that $\sigma_\alpha^\beta(y) = x$ and $\sigma_\xi^\beta(y) \in \mathcal{L}_\xi$ for all $\xi < \beta$. 

\[
\begin{align*}
 & h_\alpha \\
 & r_\alpha^0, r_\alpha^1, r_\alpha^2, \ldots, q_\alpha, X_\beta \subseteq Q^\beta \\
 & X_\alpha \cong \text{MS}
\end{align*}
\]
Recap of Open questions:

Rudin’s Question+: Is there a compact $X$ which is
(1) non-metrizable, and
(2) locally connected, and
(3) hereditarily Lindelöf (HL)?
(4) and hereditarily separable (HS)?

Question: Is there, in ZFC, an HL Aronszajn compactum?

To refute the existence of an HL Aronszajn compactum,
one needs more than just an Aronszajn tree of closed sets,
since this much exists in the Cantor set:

Proposition There is an Aronszajn tree $T$ whose nodes are closed
subsets of the Cantor set $2^\omega$. The tree ordering is $\supset$, with root $2^\omega$.
Each level of $T$ consists of a pairwise disjoint family of sets.

Proof: it’s like that of Theorem 4 of Galvin and Miller
which is attributed there to Todorčević.

Reference: F. Galvin and A. Miller, $\gamma$-sets and other singular sets of