Resolvability and almost resolvability

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BASIC DEFINITIONS:

X is κ-**resolvable** iff $\exists\kappa$ disjoint dense sets in X

X is **maximally resolvable** iff X is $\Delta(X)$-resolvable, where

$$\Delta(X) = \min\{|G| : G \neq \emptyset \text{ open in } X\}$$

κ-**irresolvable** $\equiv$ **not** κ-resolvable

X is **almost** κ-**resolvable** iff $\exists\kappa$ almost disjoint dense sets in X w.r.t. $\mathcal{N}(X)$ = ideal of nwd subsets of X

CLASSICAL, “EASY” RESULTS:

– **COMPACT, METRIC**, and **ORDERED** spaces are MAXIMALLY RESOLVABLE

Proofs are “elementary”: use BERNSTEIN’s thm

– There is a **COUNTABLE** crowded T_3 space that is (2-)IRRESOLVABLE

PROOF: take a maximal crowded T_3 extension of $\tau_{\mathbb{Q}}$

I. SEPARATION OF RESOLVABILITY PROPERTIES

DEFINITION. Fix a family $\mathcal{D}$ of dense subsets of X. $M \subset X$ is a $\mathcal{D}$-**mosaic** iff there is a maximal disjoint $\mathcal{V} \subset \tau(X)$ and for each $V \in \mathcal{V}$ there is $D_V \in \mathcal{D}$ with

$$M = \cup\{V \cap D_V : V \in \mathcal{V}\}$$

Every $\mathcal{D}$-mosaic is dense.

X is $\mathcal{D}$-**forced** iff every dense subset of X includes a $\mathcal{D}$-mosaic.

If $\mathcal{D}$ is simple and X is $\mathcal{D}$-forced we have a handle on all dense subsets of X.

$$\mathbb{B} = \{\langle B^0_\zeta, B^1_\zeta\rangle : \zeta < 2^\kappa\} \in \Pi(\kappa)$$

means that $\mathbb{B}$ is a κ-independent, (i.e.

$$|B^{i_1}_{\zeta_1} \cap B^{i_2}_{\zeta_2} \cap \ldots \cap B^{i_n}_{\zeta_n}| = \kappa$$

for $\zeta_1 < \zeta_2 < \ldots < \zeta_n < 2^\kappa$) separating family of 2-partitions of κ.

$$X_{\mathbb{B}} = \langle \kappa, \tau_{\mathbb{B}}\rangle\,,$$

where $\tau_{\mathbb{B}}$ is generated by $\{B^0_\zeta,\, B^1_\zeta : \zeta < 2^\kappa\}$.

Spaces $X_{\mathbb{B}}$ with $\mathbb{B} \in \Pi(\kappa)$ are the same as κ-dense subspaces $X \subset D(2)^{2^\kappa}$ with $|X| = \kappa$ $\equiv \mathcal{C}(\kappa)$-spaces.

THEOREM. Given $\mathbb{B} \in \Pi(\kappa)$ and a family $\mathcal{D}$ of κ-dense sets in $X_{\mathbb{B}}$, there is $\mathbb{C} \in \Pi(\kappa)$ s.t. each $D \in \mathcal{D}$ is κ-dense in $X_{\mathbb{C}}$ and $X_{\mathbb{C}}$ is $\mathcal{D}$-forced. Moreover, $X_{\mathbb{C}}$ is NODEC and $[\kappa]^{<\kappa} \subset \mathcal{N}(X_{\mathbb{C}})$.

[J-Soukup-Szentmiklóssy, $\mathcal{D}$-forced spaces: A new approach to resolvability, Top. Appl. 153 (2006), 1800–1824]
Similar results by Comfort and Hu.

PROBLEM (Ceder and Pearson, 1967) Is an ω-resolvable space maximally resolvable?

Very strong counter-examples:

THEOREM. For any $\omega < \lambda = \mathrm{cf}(\lambda) \le \kappa$ there is a $\mathcal{C}(\kappa)$-space that is hereditarily μ-resolvable $\forall\, \mu < \lambda$ but is not almost λ-resolvable.

[hereditarily μ-resolvable $\equiv$ all **crowded** subspaces are μ-resolvable]

THEOREM. Let $\omega \le \kappa < \lambda = \mathrm{cf}(\lambda) \le (2^\kappa)^+$. Then there is a $\mathcal{C}(\kappa)$-space that is

(1) hereditarily κ-resolvable,

(2) hereditarily almost μ-resolvable $\forall \mu < \lambda$,

(3) not almost λ-resolvable.

FACT. Almost ω-resolvable $\equiv$ ω-resolvable

So the following thm (for $\kappa > \omega$) also yields a solution to the Ceder–Pearson problem.

THEOREM (J-Shelah-Soukup)
For each $\kappa \geq \omega$ there is a $\mathcal{C}(\kappa)$-space that is almost 2^κ-resolvable (i.e. maximally almost resolvable) and ω_1-irresolvable.

The proof requires a further, non-trivial, refinement of a topology $\tau_{\mathbb{C}}$ obtained from the main thm.

THEOREM (Illanes, Bhaskara Rao)
If $\mathrm{cf}(\lambda) = \omega$ and a space is μ-resolvable for all $\mu < \lambda$ then it is also λ-resolvable.

PROBLEMS.

(i) What if $\omega < \mathrm{cf}(\lambda) < \lambda$?

(ii)How about singular compactness for almost resolvability?

II. TOPOLOGICAL PROPERTIES IMPLYING RESOLVABILITY

LEMMA 1. If $Y \subset X : Y$ is κ-resolvable is a π-network in X then X is κ-resolvable.

Since $\{U \in \tau(X) : |U| = \Delta(U)\}$ is a π-base, this often let's us reduce proofs to $|X| = \Delta(X)$.

LEMMA 2. If no non-empty open $U \subset X$ is covered by fewer than κ left-separated subspaces then X is κ-resolvable.

COROLLARY. Every crowded countably compact T_3 space is ω_1-resolvable.

Tkachenko (1979): If a countably compact T_3 space is σ-left-separated then it is scattered.

Apply this to all regular closed sets. (Pytkeev gave a different proof.)

PROBLEM. Are crowded countably compact T_3 spaces $\mathbf{c}$-resolvable (or even maximally resolvable)?

LEMMA 3. If D is dense in X with $|D| \leq \kappa$ and $\mathcal{I} \subset \mathcal{P}(X)$ is s.t. for any $x \in D$ and for any disjoint $\mathcal{Y} \subset [\mathcal{I}]^{<\kappa}$ there is $Z \in \mathcal{I}$ with $x \in \overline{Z}$ and $Z \cap \cup\mathcal{Y} = \emptyset$ then X is κ-resolvable.

Applying this with $\mathcal{I} = [\kappa]^{\leq\mu}$ (where $\mu < \kappa$) and $|X| = \Delta(X) = \kappa$ yields:

THEOREM. (Pytkeev)
If $\Delta(X) > t(X)$ then X is maximally resolvable.

Another application (with $\mathcal{I} = NS(\kappa)$ for $\kappa = \mathrm{cf}(\kappa) > \omega$) yields:

THEOREM. (J-S-Sz)
Any X with $\Delta(X) > s(X)$ (in particular, any HL X with $\Delta(X) > \omega$) is maximally resolvable.

With further work one gets the answer to a question of Malychin:

THEOREM. (Filatova)
Any Lindelöf T_3 space X with $\Delta(X) > \omega$ is 2-resolvable.

Recall that

COMPACT = CTBLY COMPACT + LINDELÖF

PROBLEM. Is it 3-, ω-, or maximally resolvable?

THEOREM. (Pavlov)
If X is T_3 and $\Delta(X) > e(X)^+$ then X is ω-resolvable. In particular, any Lindelöf T_3 space X with $\Delta(X) > \omega_1$ is ω-resolvable.

THEOREM (Kunen-Szymanski-Tall)
The existence of an irresolvable T_2 Baire space is equiconsistent with that of a measurable cardinal.

THEOREM (Pytkeev, 1983)
Pseudo-radial and k-spaces are both maximally resolvable.

DEFINITION.
(i) The point $p \in X$ is a B_κ-point iff it has a local π-network consisting of κ sets of size κ.
(ii) X is a Pytkeev space iff for any non-closed $A \subset X$ there is a B_κ-point $p \in \overline{A}^\kappa \setminus A$.

FACT. Pseudo-radial and k-spaces are both Pytkeev.

The first is trivial; the second uses

$$\chi(p, K) = \psi(p, K)$$

for $p \in K$ compact T_2.

THEOREM (Pytkeev)
Every Pytkeev space is maximally resolvable.

COROLLARY. (Pytkeev)
If every $p \in X$ is a B_κ-point for some $\kappa \geq \mu$ (depending on p) then X is μ-resolvable.

MONOTONE NORMALITY

X is *monotonically normal (MN)* iff it's T_1 and admits a monotone normality operator H:

For $x \in U$, $(x, U) \mapsto H(x, U)$ open s.t.

(i) $x \in H(x, U) \subset U$,

(ii) if $H(x, U) \cap H(y, V) \neq \emptyset$ then $x \in V$ or $y \in U$

FACT. Metric and ordered spaces are MN.

$D \subset X$ is *strongly discrete* iff the points in D have disjoint nbd's

FACT. In MN spaces,

discrete $\equiv$ strongly discrete

DEFINITION.
X is SD iff every non-isolated point in X is the accumulation point of a strongly discrete set

FACT. (DTTW) Every MN space is SD

HSD $\equiv$ hereditary SD

DSD $\equiv$ dense subspaces are SD

MN $\Rightarrow$ HSD $\Rightarrow$ DSD $\Rightarrow$ SD

None of these arrows can be reversed.

All results from now on are from: [J-S-Sz, Resolvability and monotone normality, Israel J. Math. 166 (2008), 1-16.]

THEOREM 1. Every crowded SD space is ω-resolvable.

THEOREM 2. Every crowded DSD space is almost $\min\{\omega_2, 2^\omega\}$-resolvable.

QUESTION. How sharp are these results (in particular, for MN spaces)?

MN topologies on trees:

$(T, <)$ a tree, for $t \in T$, $S_t =$ set of immediate successors of t in T.

T is **infinitely branching** iff S_t is infinite for all $t \in T$.

DEFINITION. Let T be infinitely branching. A map F defined on T is a **filtration** on T if $F(t)$ is a filter on S_t extending the Frechet filter on S_t. F is an **ultrafiltration** if each $F(t)$ is a (free) ultrafilter on S_t.

FACT:

$$\tau_F = \Big\{ V \subset T : \forall t \in V \ \Big(V \cap S_t \in F(t) \Big) \Big\}$$

is a MN topology on T.

We put

$$X(F) = \langle T, \tau_F \rangle$$

THEOREM 3. Let F be an ultrafiltration on T and λ be a regular cardinal such that $F(t)$ is λ-descendingly complete for all $t \in T$. Then $X(F)$ is hereditarily λ^+-irresolvable.

COROLLARY. κ measurable, u a measure on κ, $T = \mathrm{Seq}(\kappa) = \kappa^{<\omega}$, and $F(t) = u$ for all $t \in T$, then $|X(F)| = \Delta(X(F)) = \kappa$ and $X(F)$ is a hereditarily ω_1-irresolvable MN space.

FACT. (Magidor) $\exists$ supercompact $\Rightarrow$ it is consistent to have a uniform ultrafilter on $\aleph_\omega$ that is ω_1-descendingly complete.

COROLLARY. From a supercompact it is consistent to have a MN space X such that $|X| = \Delta(X) = \aleph_\omega$ and X is hereditarily ω_2-irresolvable. In particular, X is not maximally resolvable.

PROBLEM 1. Can we have a "small" MN space X with $\Delta(X) > \omega$ that is ω_1-irresolvable?

THEOREM 4. If X is MN and $|X| < \aleph_\omega$ then X is maximally resolvable.

FACT. (Kunen and Prikry, 1971) λ is regular and u is λ-desc. complete then u is λ^+-desc. complete. Consequently, if $\kappa < \aleph_\omega$ and u is a uniform ultrafilter on κ then u is λ-desc. incomplete for all $\lambda \in [\omega, \kappa]$.

THEOREM 5. If F is an ultrafiltration on Seq(κ) such that $F(t)$ is λ-desc. incomplete whenever $t \in$ Seq(κ) and $\lambda \in [\omega, \mu]$ then $X(F)$ is μ-resolvable.

COROLLARY. If $\kappa < \aleph_\omega$ and F is a uniform ultrafiltration on Seq(κ) then $X(F)$ is κ-resolvable (hence maximally resolvable).

THEOREM 6. Fix $\lambda \leq \kappa = \mathrm{cf}(\kappa)$. TFAE

(i) Every DSD space X with $|X| = \Delta(X) = \kappa$ is λ-resolvable.

(ii) Every MN space X with $|X| = \Delta(X) = \kappa$ is λ-resolvable.

(iii) For every uniform ultrafiltration F on $\mathrm{Seq}(\kappa)$, the space $X(F)$ is λ-resolvable.

COROLLARY. If X is DSD with $|X| < \aleph_\omega$ then X is maximally resolvable.

PROBLEM 2. Is there a ZFC example of a MN space that is not maximally resolvable?

NOTE. Every space of the form $X(F)$ is almost 2^ω-resolvable.

PROBLEM 3. Is every crowded MN space almost 2^ω-resolvable? (YES, if $2^\omega \leq \omega_2$.)