Randomized Models and Continuous Logic

H. Jerome Keisler
Overview

Intuitively, a randomization of a first order structure $\mathcal{M}$ is a new structure whose elements are random elements of $\mathcal{M}$. In many cases, the random elements of $\mathcal{M}$ have properties analogous to those of the original elements of $\mathcal{M}$. 

To capture this idea, we introduce the notion of a randomization of a first order theory, which is a corresponding theory in continuous logic. Continuous model theory has recently been developed in a way that looks much like first order model theory. In this lecture I will present some results showing that many model-theoretic properties of a first order theory carry over to its randomization.
Intuitively, a randomization of a first order structure $\mathcal{M}$ is a new structure whose elements are random elements of $\mathcal{M}$. In many cases, the random elements of $\mathcal{M}$ have properties analogous to those of the original elements of $\mathcal{M}$.

To capture this idea, we introduce the notion of a randomization of a first order theory, which is a corresponding theory in continuous logic. Continuous model theory has recently been developed in a way that looks much like first order model theory. In this lecture I will present some results showing that many model-theoretic properties of a first order theory carry over to its randomization.
Introduction

Some References


Randomized Models and Continuous Logic

H. Jerome Keisler

Introduction

Randomizing a Model

Given a first order vocabulary $L$ and structure $M$ with $|M| > 1$. A **randomization of** $M$ is a pair $M^R = (K, B)$ equipped with an atomless finitely additive probability space $(\Omega, B, \mu)$ such that:

- For each formula $\psi(\vec{x})$ of $L$ and tuple $\vec{X}$ in $K$, $\{w \in \Omega : M|_\psi = \psi(\vec{X}(w))\} \in B$.
- For $B, C \in B$, write $B = C$ for $\mu(B \triangle C) = 0$. Then $\forall B \exists X \exists Y B = [X = Y]$.
- For each formula $\theta(x, \vec{y})$ of $L$, $\forall \vec{Y} \exists X [\theta(X, \vec{Y})] = [\exists x \theta(\vec{Y})]$.
Randomized Models and Continuous Logic

H. Jerome Keisler

Introduction

Continuous Model Theory
Randomization Theory, $T^R$
Separable Models
Stability

Randomizing a Model

Given a first order vocabulary $L$ and structure $M$ with $|M| > 1$. A randomization of $M$ is a pair $M^R = (\mathcal{K}, B)$ equipped with an atomless finitely additive probability space $(\Omega, B, \mu)$ such that:

- $\mathcal{K}$ is a set of functions $X : \Omega \rightarrow M$. 

For each formula $\psi(\vec{x})$ of $L$ and tuple $\vec{X}$ in $K$, $\lbrack \psi(\vec{X}) \rbrack = \{ w \in \Omega : M|\psi = \psi(X(w)) \} \in B$. 

For $B, C \in B$, write $B \triangleq C$ for $\mu(B \triangle C) = 0$. Then $\forall B \exists X \exists Y B = \lbrack X = Y \rbrack$. 

For each formula $\theta(x, \vec{Y})$ of $L$, $\forall \vec{Y} \exists X \lbrack \theta(X, \vec{Y}) \rbrack \triangleq \lbrack (\exists x \theta)(\vec{Y}) \rbrack$. 

Given a first order vocabulary $L$ and structure $M$ with $|M| > 1$. A randomization of $M$ is a pair $M^R = (\mathcal{K}, \mathcal{B})$ equipped with an atomless finitely additive probability space $(\Omega, \mathcal{B}, \mu)$ such that:

- $\mathcal{K}$ is a set of functions $\mathcal{X} : \Omega \to M$.
- $\mathcal{B}$ is a set of subsets of $\Omega$, called events.
Randomizing a Model

Given a first order vocabulary $L$ and structure $M$ with $|M| > 1$. A randomization of $M$ is a pair $M^R = (\mathcal{K}, \mathcal{B})$ equipped with an atomless finitely additive probability space $(\Omega, \mathcal{B}, \mu)$ such that:

- $\mathcal{K}$ is a set of functions $X : \Omega \rightarrow M$.
- $\mathcal{B}$ is a set of subsets of $\Omega$, called events.
- For each formula $\psi(\bar{x})$ of $L$ and tuple $\bar{X}$ in $\mathcal{K}$,
  $$\llbracket \psi(\bar{X}) \rrbracket = \{ w \in \Omega : M \models \psi(\bar{X}(w)) \} \in \mathcal{B}.$$
Randomized Models and Continuous Logic

H. Jerome Keisler

Introduction

Randomizing a Model

Given a first order vocabulary \( L \) and structure \( M \) with \( |M| > 1 \). A randomization of \( M \) is a pair \( M^R = (\mathcal{K}, \mathcal{B}) \) equipped with an atomless finitely additive probability space \((\Omega, \mathcal{B}, \mu)\) such that:

- \( \mathcal{K} \) is a set of functions \( X : \Omega \to M \).
- \( \mathcal{B} \) is a set of subsets of \( \Omega \), called events.
- For each formula \( \psi(\vec{x}) \) of \( L \) and tuple \( \vec{X} \) in \( \mathcal{K} \),
  \[ \mathcal{K}[\psi(\vec{X})] = \{ w \in \Omega : M \models \psi(\vec{X}(w)) \} \in \mathcal{B}. \]
- For \( B, C \in \mathcal{B} \), write \( B \equiv C \) for \( \mu(B \triangle C) = 0 \). Then
  \[ \forall B \exists X \exists Y B \equiv [X = Y]. \]
Given a first order vocabulary $L$ and structure $M$ with $|M| > 1$. A randomization of $M$ is a pair $M^R = (\mathcal{K}, \mathcal{B})$ equipped with an atomless finitely additive probability space $(\Omega, \mathcal{B}, \mu)$ such that:

- $\mathcal{K}$ is a set of functions $X : \Omega \rightarrow M$.
- $\mathcal{B}$ is a set of subsets of $\Omega$, called events.
- For each formula $\psi(\vec{x})$ of $L$ and tuple $\vec{X}$ in $\mathcal{K}$,
  $\frown [\psi(\vec{X})] = \{ w \in \Omega : M \models \psi(\vec{X}(w)) \} \in \mathcal{B}$.
- For $B, C \in \mathcal{B}$, write $B \upharpoonright C$ for $\mu(B \triangle C) = 0$. Then
  $\forall B \exists X \exists Y \, B \upharpoonright [X = Y]$.
- For each formula $\theta(x, \vec{y})$ of $L$,
  $\forall \vec{Y} \exists X \, [\theta(X, \vec{Y})] \upharpoonright \{(\exists x \theta)(\vec{Y})\}$.
Continuous Model Theory

First Order versus Continuous Logic

- First order logic:
  Universe set $M$, equality symbol $=$.
  Functions $F : M^k \to M$, relations $P : M^k \to \{\top, \bot\}$.
  Formulas take truth values in $\{\top, \bot\}$.
  Connectives $\varphi \land \psi, \varphi \lor \psi, \neg \varphi$.
  Quantifiers $\forall, \exists$.

- Continuous logic:
  Complete metric space $N$ of diameter 1, distance $d$.
  Uniformly continuous functions and relations.
  Formulas take truth values in $[0, 1]$.
  Connectives 0, 1, $\varphi/2$, $\varphi \cdot \psi$.
  Quantifiers $\sup, \inf$.

First order structures with the discrete metric are continuous structures.
Continuous Model Theory

First Order versus Continuous Logic

- First order logic:
  Universe set $M$, equality symbol $\equiv$.
  Functions $F : M^k \to M$, relations $P : M^k \to \{\top, \bot\}$.
  Formulas take truth values in $\{\top, \bot\}$.
  Connectives $\varphi \land \psi, \varphi \lor \psi, \neg \varphi$.
  Quantifiers $\forall, \exists$.

- Continuous logic:
  Complete metric space $N$ of diameter 1, distance $d$.
  Uniformly continuous functions and relations.
  Formulas take truth values in $[0, 1]$.
  Connectives $0, 1, \varphi/2, \varphi \dot{-} \psi$.
  Quantifiers sup, inf.
First Order versus Continuous Logic

- **First order logic:**
  Universe set $M$, equality symbol $=$. Functions $F : M^k \to M$, relations $P : M^k \to \{\top, \bot\}$. Formulas take truth values in $\{\top, \bot\}$. Connectives $\varphi \land \psi, \varphi \lor \psi, \neg \varphi$. Quantifiers $\forall, \exists$.

- **Continuous logic:**
  Complete metric space $N$ of diameter 1, distance $d$. Uniformly continuous functions and relations. Formulas take truth values in $[0, 1]$. Connectives 0, 1, $\varphi/2$, $\varphi \cdot \psi$. Quantifiers sup, inf.

- First order structures with the discrete metric are continuous structures.
Vocabulary: A set of function and relation symbols.
Vocabulary: A set of function and relation symbols.

Prestructure $\mathcal{N} = (N, d, \ldots)$: Bounded pseudometric with uniformly continuous functions and relations.
Continuous Model Theory

Continuous Structures

- Vocabulary: A set of function and relation symbols.
- Prestructure $\mathcal{N} = (N, d, \ldots)$: Bounded pseudometric with uniformly continuous functions and relations.
- One-sorted example: Normed linear spaces.
- Two-sorted example: Randomizations.
Vocabulary: A set of function and relation symbols.

Prestructure $\mathcal{N} = (N, d, \ldots)$: Bounded pseudometric with uniformly continuous functions and relations.

One-sorted example: Normed linear spaces.

Two-sorted example: Randomizations.

Structure: Prestructure where $d$ is a complete metric.
Continuous Model Theory

Continuous Structures

- Vocabulary: A set of function and relation symbols.
- Prestructure $\mathcal{N} = (\mathcal{N}, d, \ldots)$: Bounded pseudometric with uniformly continuous functions and relations.
- One-sorted example: Normed linear spaces.
- Two-sorted example: Randomizations.
- Structure: Prestructure where $d$ is a complete metric.
- The completion of $\mathcal{N}$ is a structure $\hat{\mathcal{N}}$. To get $\hat{\mathcal{N}}$: identify elements of distance 0 and complete the metric.
Continuous Model Theory

Continuous Structures

- Vocabulary: A set of function and relation symbols.
- Prestructure $\mathcal{N} = (N, d, \ldots)$: Bounded pseudometric with uniformly continuous functions and relations.
- One-sorted example: Normed linear spaces.
- Two-sorted example: Randomizations.
- Structure: Prestructure where $d$ is a complete metric.
- The **completion** of $\mathcal{N}$ is a structure $\hat{\mathcal{N}}$. To get $\hat{\mathcal{N}}$: identify elements of distance 0 and complete the metric.
- $\mathcal{N} \models \varphi$ means the sentence $\varphi$ has value 0 in $\mathcal{N}$. 

Randomized Models and Continuous Logic

H. Jerome Keisler

Introduction
Continuous Model Theory
Randomization Theory, $T^R$
Separable Models
Stability
Continuous Model Theory

Continuous Structures

- Vocabulary: A set of function and relation symbols.
- Prestructure $\mathcal{N} = (N, d, \ldots)$: Bounded pseudometric with uniformly continuous functions and relations.
- One-sorted example: Normed linear spaces.
- Two-sorted example: Randomizations.
- Structure: Prestructure where $d$ is a complete metric.
- The completion of $\mathcal{N}$ is a structure $\hat{\mathcal{N}}$. To get $\hat{\mathcal{N}}$: identify elements of distance 0 and complete the metric.
- $\mathcal{N} \models \varphi$ means the sentence $\varphi$ has value 0 in $\mathcal{N}$.
- Complete theory of $\mathcal{N}$: $Th(\mathcal{N}) = \{ \varphi : \mathcal{N} \models \varphi \}$. 
Continuous Model Theory

Continuous Structures

- Vocabulary: A set of function and relation symbols.
- Prestructure $\mathcal{N} = (\mathcal{N}, d, \ldots)$: Bounded pseudometric with uniformly continuous functions and relations.
- One-sorted example: Normed linear spaces.
- Two-sorted example: Randomizations.
- Structure: Prestructure where $d$ is a complete metric.
- The **completion** of $\mathcal{N}$ is a structure $\mathcal{N}$. To get $\mathcal{N}$: identify elements of distance 0 and complete the metric.
- $\mathcal{N} \models \varphi$ means the sentence $\varphi$ has value 0 in $\mathcal{N}$.
- Complete theory of $\mathcal{N}$: $Th(\mathcal{N}) = \{ \varphi : \mathcal{N} \models \varphi \}$.
- Elementary equivalence $\mathcal{N}_1 \equiv \mathcal{N}_2$: $Th(\mathcal{N}_1) = Th(\mathcal{N}_2)$.
- For each $\mathcal{N}$, $\mathcal{N} \equiv \mathcal{N}$.
Many notions and results from first order model theory have analogues in continuous model theory. For instance:
Many notions and results from first order model theory have analogues in continuous model theory. For instance:

- Ultraproducts and Los’s theorem work for continuous logic.
Many notions and results from first order model theory have analogues in continuous model theory. For instance:

- Ultraproducts and Los’s theorem work for continuous logic.
- Compactness: Any continuous theory which is finitely satisfiable is satisfiable.
Many notions and results from first order model theory have analogues in continuous model theory. For instance:

- **Ultraproducts and Los’s theorem work for continuous logic.**
- **Compactness:** Any continuous theory which is finitely satisfiable is satisfiable.
- **The type** of an $n$-tuple $\bar{a}$ in $\mathcal{N}$ is the function $p$ mapping each formula $\varphi$ to its value $\varphi^p \in [0, 1]$ in $\mathcal{N}$ at $\bar{a}$. 

For a theory $U$, the Stone space $S_n(U)$ is the set of all types of $n$-tuples in models of $U$.

$S_n(U)$ is compact Hausdorff, with basic closed sets $\{p: \varphi^p \in C\}$ where $\varphi$ is a formula and $C$ is closed.
Continuous Model Theory

Compactness and Types

Many notions and results from first order model theory have analogues in continuous model theory. For instance:

- Ultraproducts and Los’s theorem work for continuous logic.
- Compactness: Any continuous theory which is finitely satisfiable is satisfiable.
- The type of an $n$-tuple $\bar{a}$ in $\mathcal{N}$ is the function $p$ mapping each formula $\varphi$ to its value $\varphi^p \in [0, 1]$ in $\mathcal{N}$ at $\bar{a}$.
- For a theory $U$, the Stone space $S_n(U)$ is the set of all types of $n$-tuples in models of $U$. 

Many notions and results from first order model theory have analogues in continuous model theory. For instance:

- Ultraproducts and Los’s theorem work for continuous logic.
- Compactness: Any continuous theory which is finitely satisfiable is satisfiable.
- The type of an \( n \)-tuple \( \bar{a} \) in \( \mathcal{N} \) is the function \( p \) mapping each formula \( \varphi \) to its value \( \varphi^p \in [0, 1] \) in \( \mathcal{N} \) at \( \bar{a} \).
- For a theory \( U \), the Stone space \( S_n(U) \) is the set of all types of \( n \)-tuples in models of \( U \).
- \( S_n(U) \) is compact Hausdorff, with basic closed sets \( \{ p : \varphi^p \in C \} \) where \( \varphi \) is a formula and \( C \) is closed.
The Randomization Theory $T^R$

Vocabulary

$T$ is a complete first order theory with vocabulary $L$. We always assume $T \models \exists x \exists y x \neq y$.

The randomization theory $T^R$ has vocabulary $L^R$ with:

- Two sorts, $K$ for random elements and $B$ for events.
- An $n$-ary function $\left[ \phi(\cdot) \right]$ of sort $K^n \rightarrow B$ for each first order formula $\phi(\cdot)$ with $n$ free variables.
- A unary relation $\mu$ of sort $B$ for the probability of an event.
- The Boolean operations $\top, \bot, \sqcap, \sqcup, \neg$ of sort $B$.
- Distance relations $d_K$ for sort $K$ and $d_B$ for sort $B$.
- $\forall x (\phi(x) \leq r)$ means $(\sup x \phi(x)) \leq r$.
- $\exists x (\phi(x) \leq r)$ means $(\inf x \phi(x)) \leq r$.
- $u = v$ means $d_B(u, v) = 0$. 
Vocabulary

$T$ is a complete first order theory with vocabulary $L$. We always assume $T \models \exists x \exists y x \neq y$.

The randomization theory $T^R$ has vocabulary $L^R$ with:

- Two sorts, $K$ for random elements and $B$ for events.
The Randomization Theory $T^R$

Vocabulary

$T$ is a complete first order theory with vocabulary $L$. We always assume $T \models \exists x \exists y x \neq y$.

The randomization theory $T^R$ has vocabulary $L^R$ with:

- Two sorts, $K$ for random elements and $B$ for events.
- An $n$-ary function $[\varphi(\cdot)]$ of sort $K^n \to B$ for each first order formula $\varphi(\cdot)$ with $n$ free variables.
The Randomization Theory $T^R$

Vocabulary

$T$ is a complete first order theory with vocabulary $L$. We always assume $T \models \exists x \exists y \ x \neq y$.

The randomization theory $T^R$ has vocabulary $L^R$ with:

- Two sorts, $K$ for random elements and $B$ for events.
- An $n$-ary function $\llbracket \varphi(\cdot) \rrbracket$ of sort $K^n \to B$ for each first order formula $\varphi(\cdot)$ with $n$ free variables.
- A unary relation $\mu$ of sort $B$ for the probability of an event.
The Randomization Theory $T^R$

Vocabulary

$T$ is a complete first order theory with vocabulary $L$. We always assume $T \models \exists x \exists y \, x \neq y$.
The randomization theory $T^R$ has vocabulary $L^R$ with:

- Two sorts, $K$ for random elements and $B$ for events.
- An $n$-ary function $[\varphi(\cdot)]$ of sort $K^n \to B$ for each first order formula $\varphi(\cdot)$ with $n$ free variables.
- A unary relation $\mu$ of sort $B$ for the probability of an event.
- The Boolean operations $\top, \bot, \sqcap, \sqcup, \neg$ of sort $B$. 
The Randomization Theory $T^R$

**Vocabulary**

$T$ is a complete first order theory with vocabulary $L$. We always assume $T \models \exists x \exists y x \neq y$.
The **randomization theory** $T^R$ has vocabulary $L^R$ with:

- Two sorts, $K$ for random elements and $B$ for events.
- An $n$-ary function $[\varphi(\cdot)]$ of sort $K^n \rightarrow B$ for each first order formula $\varphi(\cdot)$ with $n$ free variables.
- A unary relation $\mu$ of sort $B$ for the probability of an event.
- The Boolean operations $\top, \bot, \land, \lor, \neg$ of sort $B$.
- Distance relations $d_K$ for sort $K$ and $d_B$ for sort $B$. 

$\forall x (\varphi(x) \leq r)$ means $(\sup x \varphi(x)) \leq r$.
$\exists x (\varphi(x) \leq r)$ means $(\inf x \varphi(x)) \leq r$.
$u = v$ means $d_B(u, v) = 0$. 
The Randomization Theory $T^R$

Vocabulary

$T$ is a complete first order theory with vocabulary $L$. We always assume $T \models \exists x \exists y x \neq y$.

The randomization theory $T^R$ has vocabulary $L^R$ with:

- Two sorts, $K$ for random elements and $B$ for events.
- An $n$-ary function $[\varphi(\cdot)]$ of sort $K^n \to B$ for each first order formula $\varphi(\cdot)$ with $n$ free variables.
- A unary relation $\mu$ of sort $B$ for the probability of an event.
- The Boolean operations $\top, \bot, \cap, \cup, \neg$ of sort $B$.
- Distance relations $d_K$ for sort $K$ and $d_B$ for sort $B$.
- $\forall x (\varphi(x) \leq r)$ means $(\sup_x \varphi(x)) \leq r$.
- $\exists x (\varphi(x) \leq r)$ means $(\inf_x \varphi(x)) \leq r$.
- $u \triangleq v$ means $d_B(u, v) = 0$. 
Axioms

Validity Axioms: \( \forall \vec{x} (\lbrack \psi(\vec{x}) \rbrack \trianglerighteq \top) \)
where \( \forall \vec{x} \psi(\vec{x}) \) is logically valid in first order logic.
The Randomization Theory $T^R$

Axioms

- **Validity Axioms**: $\forall \bar{x} (\models \psi(\bar{x})) \vdash T$ 
  where $\forall \bar{x} \psi(\bar{x})$ is logically valid in first order logic.

- **Transfer Axioms**: $\models \varphi$ where $\varphi \in T$. 

The Randomization Theory $T^R$

Axioms

- Validity Axioms: $\forall \vec{x} (\models \psi(\vec{x})) \vdash T$
  where $\forall \vec{x} \psi(\vec{x})$ is logically valid in first order logic.
- Transfer Axioms: $\models \varphi \vdash T$ where $\varphi \in T$.
- The usual Boolean algebra axioms in sort $B$. 
The Randomization Theory $T^R$

**Axioms**

- **Validity Axioms**: $\forall\vec{x}(\models\psi(\vec{x})) \vdash \top$
  where $\forall\vec{x} \psi(\vec{x})$ is logically valid in first order logic.

- **Transfer Axioms**: $\models\varphi \vdash \top$ where $\varphi \in T$.

- The usual Boolean algebra axioms in sort $B$.

- **Distance Axioms**:
  $\forall x \forall y d_K(x, y) = 1 - \mu [x = y]$
  $\forall u \forall v d_B(u, v) = \mu(u \Delta v)$
The Randomization Theory $T^R$

Axioms

- Validity Axioms: $\forall \bar{x}(\llbracket \psi(\bar{x}) \rrbracket \models \top)$
  where $\forall \bar{x} \psi(\bar{x})$ is logically valid in first order logic.
- Transfer Axioms: $\llbracket \varphi \rrbracket \models \top$ where $\varphi \in T$.
- The usual Boolean algebra axioms in sort $\mathbb{B}$.
- Distance Axioms:
  $\forall x \forall y \ d_\mathcal{K}(x, y) = 1 - \mu(\llbracket x = y \rrbracket)$
  $\forall u \forall v \ d_\mathbb{B}(u, v) = \mu(u \Delta v)$
- Event Axiom: $\forall u \exists x \exists y (u \models \llbracket x = y \rrbracket)$
The Randomization Theory $T^R$

Axioms

- **Validity Axioms:** $\forall \vec{x}(\models \psi(\vec{x})) \vdash \top$
  where $\forall \vec{x} \psi(\vec{x})$ is logically valid in first order logic.

- **Transfer Axioms:** $\models \varphi \vdash \top$ where $\varphi \in T$.

- The usual Boolean algebra axioms in sort $B$.

- **Distance Axioms:**
  \[
  \forall x \forall y \ d_K(x, y) = 1 - \mu[\models x = y] \\
  \forall u \forall v \ d_B(u, v) = \mu(u \Delta v)
  \]

- **Event Axiom:** $\forall u \exists x \exists y (u \models [\models x = y])$

- **Fullness Axiom:** $\forall \vec{y} \exists x ([\models \varphi(x, \vec{y})] \models [(\exists x \varphi)(\vec{y})])$
The Randomization Theory $T^R$

**Axioms**

- **Validity Axioms:** $\forall \vec{x}(\llbracket \psi(\vec{x}) \rrbracket \models \top)$
  where $\forall \vec{x} \psi(\vec{x})$ is logically valid in first order logic.
- **Transfer Axioms:** $\llbracket \varphi \rrbracket \models \top$ where $\varphi \in T$.
- The usual Boolean algebra axioms in sort $B$.
- **Distance Axioms:**
  $\forall x \forall y \ d_K(x, y) = 1 - \mu(\llbracket x = y \rrbracket)$
  $\forall u \forall v \ d_B(u, v) = \mu(u \Delta v)$
- **Event Axiom:** $\forall u \exists x \exists y (u \models \llbracket x = y \rrbracket)$
- **Fullness Axiom:** $\forall \vec{y} \exists x (\llbracket \varphi(x, \vec{y}) \rrbracket \models \llbracket (\exists x \varphi)(\vec{y}) \rrbracket)$
- **Measure Axioms:** $\mu[\top] = 1 \land \mu[\bot] = 0$
  $\forall u \forall v (\mu(u) + \mu(v) = \mu(u \uplus v) + \mu(u \cap v))$
The Randomization Theory $T^R$

Axioms

- **Validity Axioms:** $\forall \vec{x}(\models \psi(\vec{x})) \vdash \top$
  where $\forall \vec{x} \psi(\vec{x})$ is logically valid in first order logic.

- **Transfer Axioms:** $\models \varphi \vdash \top$ where $\varphi \in T$.

- The usual Boolean algebra axioms in sort $B$.

- **Distance Axioms:**
  
  $\forall x \forall y \ d_K(x, y) = 1 - \mu[\models x = y]$
  
  $\forall u \forall \nu \ d_B(u, \nu) = \mu(u \Delta \nu)$

- **Event Axiom:** $\forall \nu \exists x \exists y (\nu \vdash [x = y])$

- **Fullness Axiom:** $\forall \vec{y} \exists x ([\models \varphi(x, \vec{y})] \vdash [\exists x \varphi(\vec{y})])$

- **Measure Axioms:**
  
  $\mu[\top] = 1 \land \mu[\bot] = 0$
  
  $\forall u \forall \nu (\mu(u) + \mu(\nu) = \mu(u \sqcup \nu) + \mu(u \sqcap \nu))$

- **Atomless Axiom:** $\forall u \exists \nu (\mu(u \sqcap \nu) = \mu(u)/2)$
The Theorem

$T^R$ is a complete theory.
The Randomization Theory $T^R$

Models

Theorem

$T^R$ is a complete theory.

Theorem

Each randomization of a model of $T$ is a premodel of $T^R$. 
The Randomization Theory $T^R$

Models

**Theorem**

$T^R$ is a complete theory.

**Theorem**

Each randomization of a model of $T$ is a premodel of $T^R$.

**Theorem**

For each $\mathcal{M} \models T$, each model of $T^R$ is isomorphic to the completion of a randomization of $\mathcal{M}$. 
The Randomization Theory $T^R$

Models

**Theorem**

$T^R$ is a complete theory.

**Theorem**

Each randomization of a model of $T$ is a premodel of $T^R$.

**Theorem**

For each $\mathcal{M} \models T$, each model of $T^R$ is isomorphic to the completion of a randomization of $\mathcal{M}$.

**Theorem**

The theory $T^R$ admits strong quantifier elimination. That is, every formula $\varphi$ in $L^R$ is $T^R$-equivalent to a formula in $L^R$ with the same free variables and no quantifiers.
The next theorem shows that we may identify types $p \in S_n(T^R)$ with regular Borel probability measures on $S_n(T)$. 
The Randomization Theory $T^R$

Types

The next theorem shows that we may identify types $p \in S_n(T^R)$ with regular Borel probability measures on $S_n(T)$.

Theorem

(i) For each type $p \in S_n(T^R)$, the function

$$\{ q \in S_n(T) : \varphi(\bar{x}) \in q \} \mapsto (\mu[\varphi(\bar{x})])^p$$

is a regular Borel probability measure $\nu_p$ on $S_n(T)$.

(ii) The mapping $p \mapsto \nu_p$ is a bijection from $S_n(T^R)$ onto the space of all regular Borel probability measures on $S_n(T)$. 
Hereafter, assume the vocabularies are countable.

**Definition**

A first order or continuous theory $U$ is $\omega$-categorical if any two separable models of $U$ are isomorphic.
Hereafter, assume the vocabularies are countable.

**Definition**

A first order or continuous theory $U$ is $\omega$-**categorical** if any two separable models of $U$ are isomorphic.

**Theorem**

$T$ is $\omega$-categorical if and only if $T^R$ is $\omega$-categorical.
Given a structure $\mathcal{N}$ and subset $A \subseteq N$, $\mathcal{N}_A$ is $\mathcal{N}$ with extra constants for the elements of $A$.

By an $n$-type over $\mathcal{N}_A$ we mean an element of $S_n(Th(\mathcal{N}_A))$, i.e., the type of an $n$-tuple in some model of $Th(\mathcal{N}_A)$.

**Definition**

$\mathcal{N}$ is $\omega$-saturated if for each finite $A \subseteq N$, every type over $\mathcal{N}_A$ is realized in $\mathcal{N}$. 
Given a structure $\mathcal{N}$ and subset $A \subseteq N$, $\mathcal{N}_A$ is $\mathcal{N}$ with extra constants for the elements of $A$.
By an **$n$-type over** $\mathcal{N}_A$ we mean an element of $S_n(Th(\mathcal{N}_A))$, i.e., the type of an $n$-tuple in some model of $Th(\mathcal{N}_A)$.

**Definition**

$\mathcal{N}$ is **$\omega$-saturated** if for each finite $A \subseteq N$, every type over $\mathcal{N}_A$ is realized in $\mathcal{N}$.

**Theorem**

$T$ has a countable $\omega$-saturated model if and only if $T^R$ has a separable $\omega$-saturated model.
Definition

$\mathcal{N}$ is elementarily embeddable in $\mathcal{N}'$ if there is a map $h : \mathcal{N} \to \mathcal{N}'$ preserving the truth value of every formula. $\mathcal{N}$ is prime if it is elementarily embeddable in every $\mathcal{N}' \equiv \mathcal{N}$.

Every prime model is separable.
**Definition**

\( \mathcal{N} \) is **elementarily embeddable** in \( \mathcal{N}' \) if there is a map \( h : \mathcal{N} \to \mathcal{N}' \) preserving the truth value of every formula. 

\( \mathcal{N} \) is **prime** if it is elementarily embeddable in every \( \mathcal{N}' \equiv \mathcal{N} \).

Every prime model is separable.

**Theorem**

\( T \) has a prime model if and only if \( T^R \) has a prime model.
ω-Stable Theories

Definition
A first order or continuous theory $U$ is $\omega$-stable if for every model $\mathcal{N}$ of $U$ and countable set $A \subseteq N$, $Th(\mathcal{N}_A)$ has a separable model which realizes every type over $\mathcal{N}_A$. 

Theorem
$T$ is $\omega$-stable if and only if $T^R$ is $\omega$-stable.
ω-Stable Theories

Definition
A first order or continuous theory $U$ is $\omega$-stable if for every model $\mathcal{N}$ of $U$ and countable set $A \subseteq N$, $Th(\mathcal{N}_A)$ has a separable model which realizes every type over $\mathcal{N}_A$.

Theorem
$T$ is $\omega$-stable if and only if $T^R$ is $\omega$-stable.
Let $\lambda$ be an infinite cardinal.

**Definition**

A first order or continuous theory $U$ is $\lambda$-**stable** if for every model $\mathcal{N}$ of $U$ and set $A \subseteq N$ of cardinality $\lambda$, $Th(\mathcal{N}_A)$ has a model of density $\lambda$ that realizes every type over $\mathcal{N}_A$.

$U$ is **stable** if $U$ is $\lambda$-stable for some $\lambda$.
Stability

Stable Theories

Let \( \lambda \) be an infinite cardinal.

**Definition**

A first order or continuous theory \( U \) is \( \lambda \)-**stable** if for every model \( N \) of \( U \) and set \( A \subseteq N \) of cardinality \( \lambda \), \( Th(N_A) \) has a model of density \( \lambda \) that realizes every type over \( N_A \).

\( U \) is **stable** if \( U \) is \( \lambda \)-stable for some \( \lambda \).

**Theorem (Ben Yaacov)**

\( T \) is stable if and only if \( T^R \) is stable.
Let $\lambda$ be an infinite cardinal.

**Definition**

A first order or continuous theory $U$ is $\lambda$-**stable** if for every model $N$ of $U$ and set $A \subseteq N$ of cardinality $\lambda$, $Th(N_A)$ has a model of density $\lambda$ that realizes every type over $N_A$.

$U$ is **stable** if $U$ is $\lambda$-stable for some $\lambda$.

**Theorem (Ben Yaacov)**

$T$ is stable if and only if $T^R$ is stable.

**Theorem (Ben Yaacov)**

$T$ is dependent if and only if $T^R$ is dependent.
Problem

Which properties of $T$ carry over to $T^R$?
Problem

Which properties of $T$ carry over to $T^R$?

Definition

$U$ is superstable if $U$ is $\lambda$-stable for all $\lambda \geq 2^\omega$. 
Problem

Which properties of $T$ carry over to $T^R$?

Definition

$U$ is **superstable** if $U$ is $\lambda$-stable for all $\lambda \geq 2^\omega$.

Problem

If $T$ is superstable, must $T^R$ be superstable?
Problem

Which properties of $T$ carry over to $T^R$?

Definition

$U$ is superstable if $U$ is $\lambda$-stable for all $\lambda \geq 2^{\omega}$.

Problem

If $T$ is superstable, must $T^R$ be superstable?

Thanks for listening!