

Randomized Models and Continuous Logic

H. Jerome Keisler

Randomized
Models and
Continuous
Logic

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Introduction

Continuous
Model Theory

Randomization
Theory, T^R

Separable
Models

Stability

Intuitively, a randomization of a first order structure \mathcal{M} is a new structure whose elements are random elements of \mathcal{M} . In many cases, the random elements of \mathcal{M} have properties analogous to those of the original elements of \mathcal{M} .

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To capture this idea, we introduce the notion of a randomization of a first order theory, which is a corresponding theory in continuous logic. Continuous model theory has recently been developed in a way that looks much like first order model theory. In this lecture I will present some results showing that many model-theoretic properties of a first order theory carry over to its randomization.

Some References

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- 5 I. Ben Yaacov and H. J. Keisler. Randomizations as Metric Structures, Submitted.

Randomizing a Model

Given a first order vocabulary L and structure \mathcal{M} with $|M| > 1$. A **randomization of \mathcal{M}** is a pair $\mathcal{M}^R = (\mathcal{K}, \mathcal{B})$ equipped with an atomless finitely additive probability space $(\Omega, \mathcal{B}, \mu)$ such that:

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- For each formula $\psi(\vec{x})$ of L and tuple \vec{X} in \mathcal{K} ,

$$\llbracket \psi(\vec{X}) \rrbracket = \{w \in \Omega : \mathcal{M} \models \psi(\vec{X}(w))\} \in \mathcal{B}.$$

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- For each formula $\theta(x, \vec{y})$ of L ,

$$\forall \vec{Y} \exists X \llbracket \theta(X, \vec{Y}) \rrbracket \doteq \llbracket (\exists x \theta)(\vec{Y}) \rrbracket.$$

First Order versus Continuous Logic

- First order logic:

Universe set M , equality symbol $=$.

Functions $F : M^k \rightarrow M$, relations $P : M^k \rightarrow \{\top, \perp\}$.

Formulas take truth values in $\{\top, \perp\}$.

Connectives $\varphi \wedge \psi, \varphi \vee \psi, \neg\varphi$.

Quantifiers \forall, \exists .

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- Continuous logic:
Complete metric space N of diameter 1, distance d .
Uniformly continuous functions and relations.
Formulas take truth values in $[0, 1]$
Connectives $0, 1, \varphi/2, \varphi \dot{-} \psi$.
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- First order structures with the discrete metric are continuous structures.

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- Elementary equivalence $\mathcal{N}_1 \equiv \mathcal{N}_2$: $Th(\mathcal{N}_1) = Th(\mathcal{N}_2)$.
- For each \mathcal{N} , $\widehat{\widehat{\mathcal{N}}} \equiv \mathcal{N}$.

Compactness and Types

Many notions and results from first order model theory have analogues in continuous model theory. For instance:

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- For a theory U , the Stone space $S_n(U)$ is the set of all types of n -tuples in models of U .
- $S_n(U)$ is compact Hausdorff, with basic closed sets $\{p : \varphi^p \in C\}$ where φ is a formula and C is closed.

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T is a complete first order theory with vocabulary L .

We always assume $T \models \exists x \exists y x \neq y$.

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- Distance relations $d_{\mathbf{K}}$ for sort \mathbf{K} and $d_{\mathbf{B}}$ for sort \mathbf{B} .
- $\forall x(\varphi(x) \leq r)$ means $(\sup_x \varphi(x)) \leq r$.
- $\exists x(\varphi(x) \leq r)$ means $(\inf_x \varphi(x)) \leq r$.
- $u \doteq v$ means $d_{\mathbf{B}}(u, v) = 0$.

Axioms

- **Validity Axioms:** $\forall \vec{x}(\llbracket \psi(\vec{x}) \rrbracket \doteq \top)$
where $\forall \vec{x} \psi(\vec{x})$ is logically valid in first order logic.

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$$\forall x \forall y d_{\mathbf{K}}(x, y) = 1 - \mu \llbracket x = y \rrbracket$$
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- Measure Axioms: $\mu[\top] = 1 \wedge \mu[\perp] = 0,$

$$\forall u \forall v (\mu(u) + \mu(v) = \mu(u \sqcup v) + \mu(u \sqcap v))$$

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- Atomless Axiom: $\forall u \exists v (\mu(u \sqcap v) = \mu(u)/2)$

The Randomization Theory T^R

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Theorem

T^R is a complete theory.

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For each $\mathcal{M} \models T$, each model of T^R is isomorphic to the completion of a randomization of \mathcal{M} .

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Theorem

The theory T^R admits strong quantifier elimination. That is, every formula φ in L^R is T^R -equivalent to a formula in L^R with the same free variables and no quantifiers.

Types

The next theorem shows that we may identify types $p \in S_n(T^R)$ with regular Borel probability measures on $S_n(T)$.

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Theorem

(i) For each type $p \in S_n(T^R)$, the function

$$\{q \in S_n(T) : \varphi(\vec{x}) \in q\} \mapsto (\mu[\varphi(\vec{x})])^p$$

is a regular Borel probability measure ν_p on $S_n(T)$.

(ii) The mapping $p \mapsto \nu_p$ is a bijection from $S_n(T^R)$ onto the space of all regular Borel probability measures on $S_n(T)$.

Hereafter, assume the vocabularies are countable.

Definition

A first order or continuous theory U is ω -**categorical** if any two separable models of U are isomorphic.

ω -Categorical Theories

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Theorem

T is ω -categorical if and only if T^R is ω -categorical.

ω -Saturated models

Given a structure \mathcal{N} and subset $A \subseteq N$, \mathcal{N}_A is \mathcal{N} with extra constants for the elements of A .

By an **n -type over \mathcal{N}_A** we mean an element of $S_n(\text{Th}(\mathcal{N}_A))$, i.e., the type of an n -tuple in some model of $\text{Th}(\mathcal{N}_A)$.

Definition

\mathcal{N} is **ω -saturated** if for each finite $A \subseteq N$, every type over \mathcal{N}_A is realized in \mathcal{N} .

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Theorem

T has a countable ω -saturated model if and only if T^R has a separable ω -saturated model.

Prime Models

Definition

\mathcal{N} is **elementarily embeddable** in \mathcal{N}' if there is a map $h : \mathcal{N} \rightarrow \mathcal{N}'$ preserving the truth value of every formula.
 \mathcal{N} is **prime** if it is elementarily embeddable in every $\mathcal{N}' \equiv \mathcal{N}$.

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A first order or continuous theory U is ω -**stable** if for every model \mathcal{N} of U and countable set $A \subseteq N$, $Th(\mathcal{N}_A)$ has a separable model which realizes every type over \mathcal{N}_A .

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Stable Theories

Let λ be an infinite cardinal.

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A first order or continuous theory U is λ -**stable** if for every model \mathcal{N} of U and set $A \subseteq N$ of cardinality λ , $Th(\mathcal{N}_A)$ has a model of density λ that realizes every type over \mathcal{N}_A .

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Theorem (Ben Yaacov)

T is stable if and only if T^R is stable.

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U is **stable** if U is λ -stable for some λ .

Theorem (Ben Yaacov)

T is stable if and only if T^R is stable.

Theorem (Ben Yaacov)

T is dependent if and only if T^R is dependent.

Problem

Which properties of T carry over to T^R ?

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Definition

U is **superstable** if U is λ -stable for all $\lambda \geq 2^\omega$.

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If T is superstable, must T^R be superstable?

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Definition

U is **superstable** if U is λ -stable for all $\lambda \geq 2^\omega$.

Problem

If T is superstable, must T^R be superstable?

Thanks for listening!