

# Set-Theoretic Geometry

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## *Outline of Talk*

1. The Filippov space under CH
  2. Its demise under PFA and its survival under MA
  3. Smooth arcs in the plane
  4. Some questions
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All spaces are  $T_3$  (Hausdorff and regular); usually they're compact.

## *References*

Some recent papers (in order of date):

- [1] K. Kunen, Locally Connected HL Compacta  
preprint available.
- [2] J. Hart & K. Kunen, Arcs in the Plane,  
preprint available, soon . . . . .
- [3] K. Kunen, Some Flavors of OCA  
preprint available, eventually, maybe . . . . .

These have references to the work of the many other people mentioned.

See my home page

<http://www.math.wisc.edu/~kunen/>

## ***Quick Overview of Topics (1)(2)***

Possible Question PQ (never “officially” asked):

Is there a compact *locally connected* hereditarily Lindelöf (HL) space which is not metrizable (= not second countable)?  
Maybe hereditarily separable (HS) also?

Possible reason for asking

The double arrow space DAS (Alexandroff and Urysohn 1929) is compact, HL, HS, and not metrizable, but totally disconnected. It’s also a compact LOTS: - - - - -

PQ asks: Is there a *locally* connected version of the DAS?

The cone over DAS is connected, but not locally connected.

Filippov 1969: Answer to PQ is yes under CH.

His space is also HS — it’s a “2D” version of DAS.

He used a Luzin (1914) set, which is refuted by  $MA + \neg CH$ , so

Published Question (1982) P. Nyikos and/or M. E. Rudin: does

$MA + \neg CH \Rightarrow$  the answer to PQ is “no”?

Note:  $HS = HL$  now (Juhász and Szentmiklóssy).

My answer: no —  $MA + \neg CH \not\Rightarrow$  the answer to PQ is “no”.

The Filippov construction succeeds

iff there is a *weakly Luzin set*.

Weakly Luzin sets are consistent with  $MA + \neg CH$

but refuted by PFA.

but PQ is still open under PFA.

Some background



## ***HS and HL***

H = hereditarily    S = separable    L = Lindelöf.

HS means all subspaces are separable.

If  $X$  is  $2^{\text{nd}}$  countable (= separable metric):

then all subspaces of  $X$  are  $2^{\text{nd}}$  countable,

and hence separable and Lindelöf,

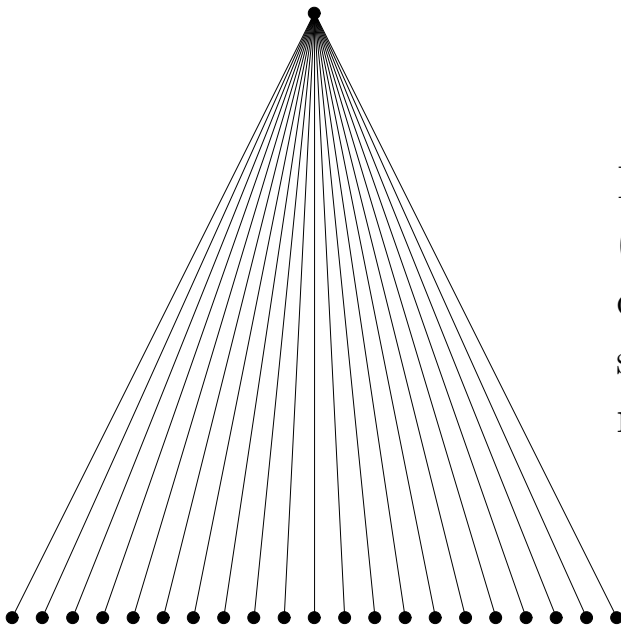
so  $X$  is HS and HL.

The DAS is compact and HS and HL and not  $2^{\text{nd}}$  countable.

It's also a compact LOTS.

The DAS is totally disconnected.

The cone over DAS is connected, but not locally connected.



Points

(other than the top one)

don't have arbitrarily

small connected

neighborhoods.

PQ was: Is there a compact *locally connected* HL (and HS) space which is not  $2^{\text{nd}}$  countable?

## The Double Arrow Space

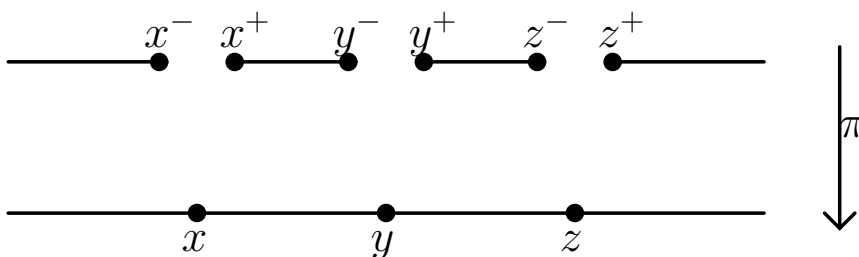
Basic construction:

Start with  $E \subseteq (0, 1) \subseteq [0, 1]$ .

Get space  $\Phi_E$  from  $[0, 1]$  by replacing each  $x \in E$   
by a pair of neighboring points,  $x^-, x^+$ .

Natural map  $\pi_E : \Phi_E \twoheadrightarrow [0, 1]$ .

Picture for  $E = \{x, y, z\}$ :



Observe (exercise):

1.  $\Phi_E$  is always compact and HS and HL.
2.  $\Phi_E$  is 2<sup>nd</sup> countable iff  $|E| \leq \aleph_0$ .
3.  $\Phi_E$  is never locally connected when  $E$  is infinite.

So let  $|E| \geq \aleph_1$

The official DAS has  $E = (0, 1)$ .

Hint for (1): use the equivalent characterizations:

$X$  is HS iff  $X$  has *no* left-separated  $\omega_1$ -sequence.

$X$  is HL iff  $X$  has *no* right-separated  $\omega_1$ -sequence.

The sequence  $\langle x_\alpha : \alpha < \omega_1 \rangle$  is:

- left separated iff each  $x_\alpha \notin \text{cl}\{x_\xi : \xi < \alpha\}$  *bad*
- right separated iff each  $x_\alpha \notin \text{cl}\{x_\xi : \xi > \alpha\}$  *bad*
- discrete iff each  $x_\alpha \notin \text{cl}\{x_\xi : \xi \neq \alpha\}$  *worse*

The Filippov space: change  $[0, 1]$  to  $[0, 1]^2$  (or  $[0, 1]^n$ ),  
change *pairs* of points  $S^0$  to *circles*  $S^1$  (or  $S^{n-1}$ ).

Then the space is locally connected.

## Пространство Филиппова

Basic construction:

Start with  $E \subseteq (0, 1)^2 \subseteq [0, 1]^2$ .

Get space  $\Phi_E$  from  $[0, 1]^2$  by replacing each  $x \in E$   
by a copy of a circle  $C_x$ .

Natural map  $\pi_E : \Phi_E \rightarrow [0, 1]^2$  identifies  
 $C_x$  back to  $x$ .

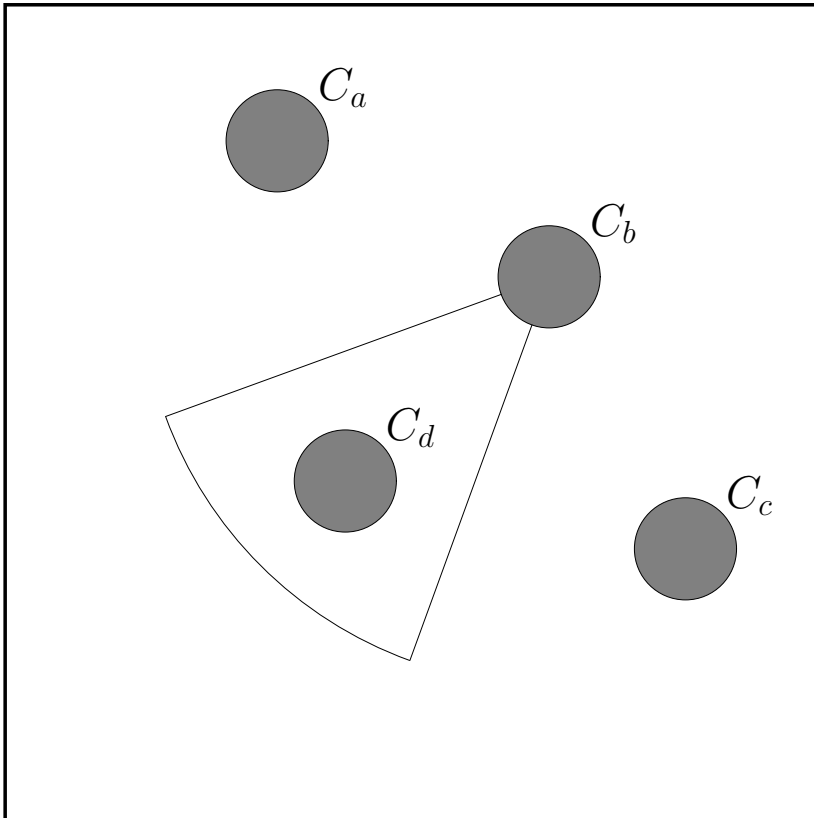
Topology:

$\pi^{-1}(U)$  is open for all open  $U \subseteq [0, 1]^2$ .

Also,  $N(x, \varepsilon, \alpha, \beta)$  are open for  $x \in E$ ,  $r > 0$ ,  $\alpha, \beta$  are “angles”.

Contains: points on the arc  $(\alpha, \beta)$  of  $C_x$   
plus points/circles at distance  $< \varepsilon$  from  $x$   
in directions lying in  $(\alpha, \beta)$ .

Picture for  $E = \{a, b, c, d\}$ ;  $N(b, 1/2, 200^\circ, 250^\circ)$  shown:



The  $C_x$  are *circles*  
not discs.

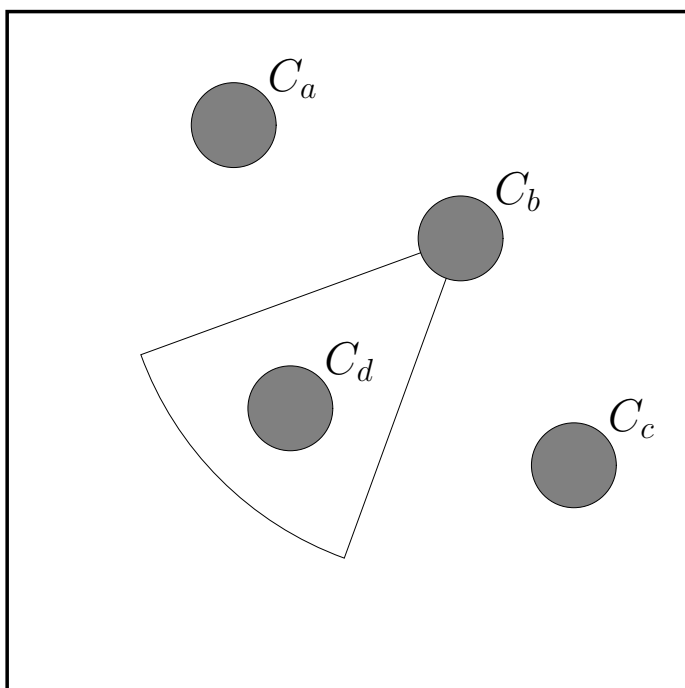
The insides are *gone*

The radii are

“infinitesimal”

Only point  $a$  gets  
replaced by  $C_a$ .

## More Filippov space



Observe (exercise)

1.  $\Phi_E$  is always compact and *rarely* HS or HL.
2.  $\Phi_E$  is  $2^{\text{nd}}$  countable iff  $|E| \leq \aleph_0$ .
3.  $\Phi_E$  is *always* connected and locally connected.

Problem with HS/HL:

If  $E$  has  $\aleph_1$  points like  $a, b, c$  (going roughly NW–SE) then there will be an uncountable discrete subset

Choose a point on the SW part of each circle.

Filippov's solution: Let  $E$  be a Luzin set.

Then  $\Phi_E$  is HS and HL — so, all is well under CH.

Problem: What if there are no Luzin sets (say, under  $\text{MA}(\aleph_1)$ )?

My solution: Let  $E$  be a *weakly* Luzin set.

This is consistent with  $\text{MA}(\aleph_1)$ .

Then  $\Phi_E$  is still HS and HL.

But, under PFA (or just SOCA) the construction fails:

Every uncountable  $E \subseteq \mathbb{R}^2$  has  $\aleph_1$  points

lined up roughly in *some* direction, as are  $a, b, c$ .

## Weakly Luzin Sets

Fix an uncountable  $E \subseteq \mathbb{R}^2$ . Then  $E$  is

Luzin iff all nowhere dense subsets of  $E$  are countable.

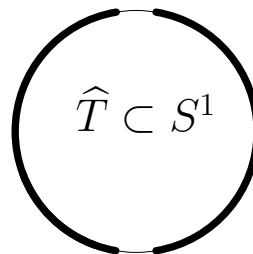
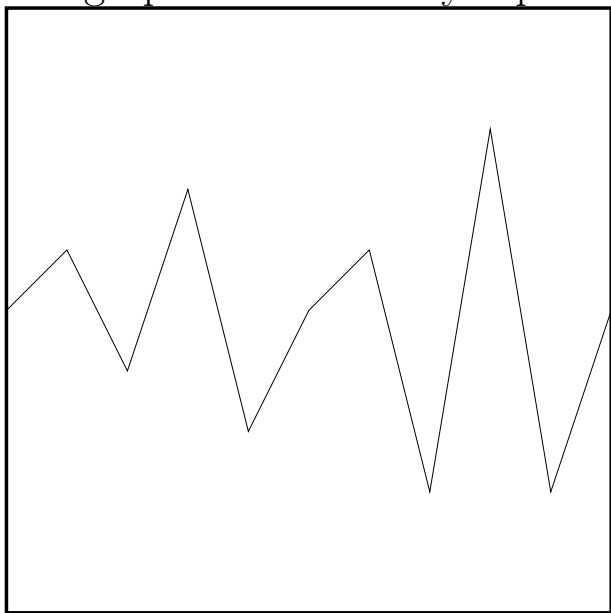
weakly Luzin iff all *skinny* subsets of  $E$  are countable.

$T$  *skinny* iff the set of “directions” between pairs of points:

$\widehat{T} := \{(x - y)/\|x - y\| : \{x, y\} \in [T]^2\}$  is  
not dense in the unit circle.

Equivalently, rotating  $T$  so that  $(0, 1) \notin \text{cl}(\widehat{T})$ ,

$T \subseteq$  the graph of a uniformly Lipschitz function.



Skinny sets are nowhere dense, so Luzin sets are weakly Luzin.

But weakly Luzin sets are consistent with  $\text{MA}(\aleph_1)$ .

*Theorem*(ZFC).  $E$  is weakly Luzin iff

$\Phi_E$  is HS iff  $\Phi_E$  is HL iff

$\Phi_E$  has no uncountable discrete subsets.

Under PFA (or just SOCA), every uncountable  $E \subseteq \mathbb{R}^2$

has uncountable *very* skinny subsets.

very skinny ?????

## Very Skinny Sets

Under PFA (or just SOCA), every uncountable  $E \subseteq \mathbb{R}^2$  has uncountable *very* skinny subsets.

For  $0 \leq \varepsilon \leq \pi/2$ , call  $T \subseteq \mathbb{R}^2$   $\varepsilon$ -directed iff for some “direction”  $v \in S^1$ :

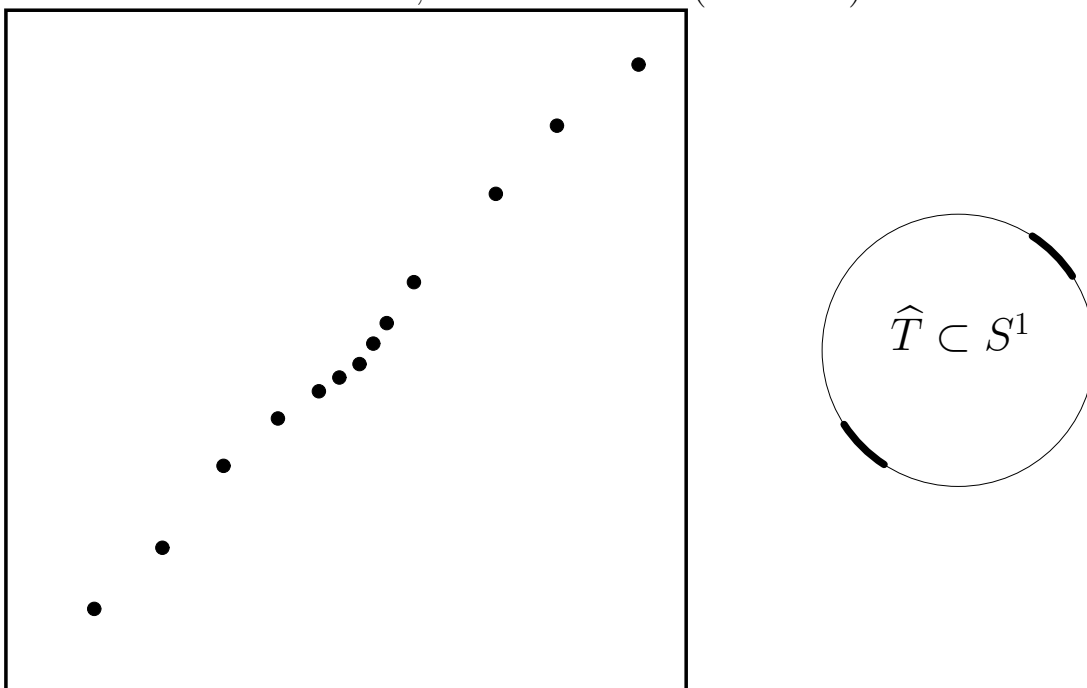
$$\angle(\pm v, (x - y)/\|x - y\|) \leq \varepsilon \quad \forall \{x, y\} \in [T]^2.$$

Every  $T$  is  $\pi/2$ -directed.  $T$  is 0-directed iff  $T \subseteq$  some straight line.

$T$  is skinny iff  $T$  is  $\varepsilon$ -directed for some  $\varepsilon < \pi/2$ .

SOCA implies  $\forall E \in [\mathbb{R}^2]^{\aleph_1} \forall \varepsilon > 0 \exists T \in [E]^{\aleph_1} [T \text{ is } \varepsilon\text{-directed}]$ .

Picture of  $12^\circ$ -directed, with  $v = 45^\circ$  (NE-SW).



Connecting the dots, we see that  $T$  lies on a rectifiable arc:

$$\text{length} \leq \sqrt{2}/\cos(12^\circ) \quad (\text{if the picture is in the unit square})$$

But the arc need not be smooth — it can have angles.

Even *better*, we can get a smooth ( $C^1$ ) arc  $A$

which meets  $E$  in an uncountable set.

Then  $\forall \varepsilon > 0$ ,  $A$  is a finite union of  $\varepsilon$ -directed arcs.

We need to remove some dots . . . . .

## *Smooth Arcs*

Definitions:

“arc” = “homeomorphic to an interval”.

The arc  $A$  is  $C^k$  ( $k \geq 1$ ) iff  $A$  is rectifiable *and* if  $g : [0, b] \rightarrow A$  is a homeomorphism using arc length as a parameter, then  $g \in C^k([0, b], \mathbb{R}^2)$ .

Theorem (PFA). Every uncountable  $E \subseteq \mathbb{R}^2$  meets some  $C^1$  arc in an uncountable set.

A weakly Luzin set is a counter-example consistent with  $\text{MA}(\aleph_1)$ .

The theorem is false in ZFC for  $C^2$ ; there is a perfect  $E \subset \mathbb{R}^2$  which meets each  $C^2$  arc in a finite set.

Technical Question. Is SOCA sufficient?

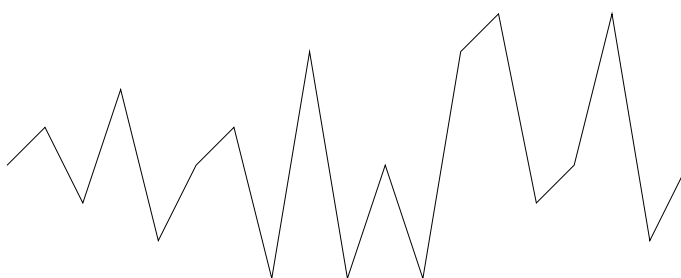
The proof needs  $\text{MA}(\aleph_1)$  (or  $\mathfrak{p} > \aleph_1$ ) plus  $\text{OCA}_{[\text{ARS}]}$ .

## Getting Smooth Arcs

Theorem (PFA). Every uncountable  $E \subseteq \mathbb{R}^2$  meets some  $C^1$  arc in an uncountable set.

Proof in three steps.

1. By SOCA and deleting some points and rotating coordinates, WLOG  $E \subset L$ , where  $L$  is (the graph of) a function on  $[0, 1]$  and is uniformly Lipschitz:  $|L(x) - L(y)|/|x - y| \leq 6$ .



2. Apply PFA to get a “thin” compact set  $Q \subseteq \mathbb{R}$  and continuous  $g : Q \rightarrow \mathbb{R}$  such that, setting  $f = L \upharpoonright Q$ :
- a.  $f \cap E$  is uncountable.
  - b.  $f' = g$  in the sense that  $(f(x_1) - f(x_2))/(x_1 - x_2) \rightarrow g(x)$  uniformly as  $x_1, x_2 \rightarrow x$  ( $x_1, x_2, x \in Q$ ).
3. Following Hermite (1901), extend  $f, g$  to  $\bar{f} \in C^1(\mathbb{R})$  and  $\bar{g} \in C(\mathbb{R})$  with  $\bar{f}' = \bar{g}$ ; so  $\bar{f}$  is the desired arc.

More on (2): The general fact (under PFA) is:

Let  $X$  be a Polish space,  $K$  a compact metric space,

$E$  an uncountable subset of  $X$  and  $h \in C([X]^2, K)$ .

Then there is a compact set  $Q \subseteq X$  such that  $Q \cap E$  is uncountable and  $h \upharpoonright [Q]^2$  extends to some  $\tilde{h} \in C(Q \times Q, K)$ .

Here,  $X = [0, 1]$ ,  $K = [-6, 6]$

and  $h(\{x, y\}) = (L(x) - L(y))/(x - y)$ .

Then  $f(x) = L(x)$  and  $g(x) = \tilde{h}(x, x)$  (for  $x \in Q$ ).

## *Questions and Minor Improvements*

Thm(PFA). Let  $X$  be a Polish space,  $K$  a compact metric space,  
 $E$  an uncountable subset of  $X$  and  $h \in C([X]^2, K)$ .

Then there is a compact set  $Q \subseteq X$  such that  $Q \cap E$  is uncountable  
and  $h \upharpoonright [Q]^2$  extends to some  $\tilde{h} \in C(Q \times Q, K)$ .

Minor improvements, with similar but somewhat more tedious proofs.

In this Theorem: Actually, if  $|E| = \aleph_1$ ,  
you can cover  $E$  by  $\aleph_0$  such compact sets,  $Q_n$  (for  $n \in \omega$ )  
(with corresponding  $\tilde{h}_n$ ).

In the last theorem:

Theorem (PFA). Every uncountable  $E \subseteq \mathbb{R}^2$  meets some  
 $C^1$  arc in an uncountable set.

Actually, if  $|E| = \aleph_1$ , then you can cover  $E$   
by  $\aleph_0$  such arcs.

Two general questions:

Question 1: Is there something general behind the “actually”?

Question 2: Can you derive such results directly from their (easier)  
ZFC analogs?

## Question 1

Assume, say, PFA.

Assume that all uncountable sets  $E$

meet a “nice” set in an uncountable set. (A)

Then when  $|E| = \aleph_1$ , can we cover  $E$  by  $\aleph_0$  “nice” sets? (B)

Example 1 (MA( $\aleph_1$ )). Yes

In the plane, “nice” = “a Cantor set”.

Example 2 (ZFC). No

In  $\omega_1$ , “nice” = “nonstationary”.

Example 3 (PFA). Yes

In the plane, “nice” = “a  $C^1$  arc”.

Example 4 (PFA). Yes

$X$  is a separable metric space,  $[X]^2 = W_0 \cup W_1$ ;  $W_0, W_1$  open

Consider  $E \subseteq X$ . “nice” is “ $W_0$ -connected or  $W_1$ -connected”.

(A) follows from SOCA but (B) is OCA<sub>[ARS]</sub>.

Example 5 (ZFC). No (the actual SOCA)

Like (4), but  $W_0$  is open and  $W_1$  is closed.

There’s a ZFC example where (A) is true and (B) is false.

Example 6 (PFA, say). I don’t know.

$X$  is a separable metric space,  $[X]^3 = W_0 \cup W_1$ ;  $W_0, W_1$  open.

“nice” is “ $W_0$ -homogeneous or  $W_1$ -homogeneous”.

(A) *could* fail (Blass, 1981), but *assume* it holds  $\forall E \subseteq X$ .

Then does (B) hold?

Question 1 is: is there something more general going on here?

## Question 2

*Remark.* “perfect set” versions of PFA partition results are often ZFC theorems and easier to prove.

Assume, say, PFA. Let  $X$  be a Polish space.

Assume that for all perfect  $P \subseteq X$

there is a “nice” perfect  $Q \subseteq P$  (C)

Then do *all* uncountable sets

meet a “nice” perfect set in an uncountable set? (A)

Example 1 (MA( $\aleph_1$ )). Yes. In the plane, “nice” = “a Cantor set”.

(C): by Cantor, essentially.

(A): see our January 2008 logic qual.

Example 3 (PFA). Yes. In the plane, “nice” = “a  $C^1$  arc”.

(C) is fairly easy.

Example 5 (PFA). Yes.

$[X]^2 = W_0 \cup W_1$ ;  $W_0$  is open and  $W_1$  is closed.

“nice” is “ $W_0$ -connected or  $W_1$ -connected”.

(C) is from Galvin (1968) ; (A) is SOCA.

Example 7 (ZFC). No.

$[X]^2 = W_0 \cup W_1$  ;  $W_0, W_1$  Borel.

“nice” is “ $W_0$ -connected or  $W_1$ -connected”.

(C) is by Mycielski (1964) + Galvin (1968)

$\neg$ (A) is by Sierpiński (1933) (  $\omega_1 \not\rightarrow (\omega_1)_2^2$  )

+ Rothberger (1944)

Question 2 is: Can we use PFA once to show that the hard thing always follows from the easy thing (when it does)?

## Remarks on Arcs

I already said:

Theorem (PFA). Every uncountable  $E \subseteq \mathbb{R}^2$  meets some  $C^1$  arc in an uncountable set.

This is false in ZFC for  $C^2$ ; there is a perfect  $E \subset \mathbb{R}^2$  which meets each  $C^2$  arc in a finite set.

MA( $\aleph_1$ ) isn't sufficient; let  $E$  be weakly Luzin.

Are the definitions right?

“arc” always means: a set  $A$  homeomorphic to an interval:

$A = g([a, b])$ , where  $g : [a, b] \rightarrow \mathbb{R}^2$  is 1-1 and continuous.

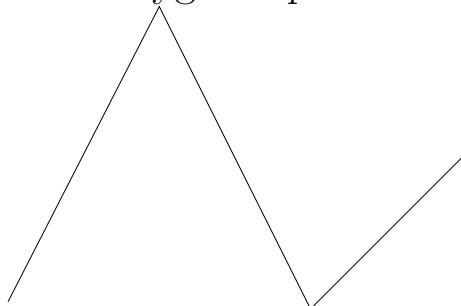
Then  $A$  is  $C^k$  ( $k \geq 1$ ) iff  $g$  can be chosen so that  $g \in C^k([a, b], \mathbb{R}^2)$  and  $g'(t) \neq 0$  for all  $t$ ;

equivalently, it's  $C^k$  using arc length as a parameter.

Why the “ $g'(t) \neq 0$ ” ? If we drop it, call  $A$  *weakly*  $C^k$ .

$C^k$  captures the geometric notion of “smooth – no corners”.

Polygonal paths are weakly  $C^\infty$



Slow down to 0 at the corners.

Use functions like  $e^{-1/t^2}$ .

“weakly  $C^k$ ” captures the physical notion of “smooth ride”.

The theorems change:

Theorem (MA( $\aleph_1$ )). Every uncountable  $E \subseteq \mathbb{R}^2$  meets some weakly  $C^\infty$  arc in an uncountable set.

actually, is covered by  $\aleph_0$  such arcs if  $|E| = \aleph_1$ .

## *Remarks on PQ*

PQ is now: Is it consistent (perhaps following from PFA) that every compact locally connected HL (equivalently, HS) space is  $2^{\text{nd}}$  countable?

It doesn't follow just from  $\text{MA}(\aleph_1)$ .

The answer would be “yes” if the answer to Fremlin's Question is “yes”:

FQ: Is it consistent (perhaps following from PFA) that every compact HL (and hence HS) space has a 2-1 map onto a compact metric space?

The DAS and the cone over the DAS (which is connected) have such maps, but a *locally* connected HL compactum with such a map (even zero-dimensional-to-one) must be compact metric itself.