How many Measures can there be?

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The Beginning: Kunen weighs in.

An answer: My contribution.

What about a model in which κ is the only measurable cardinal?

Are all those measurable cardinals needed to start with?

What is a 'measure'?

Definition

A cardinal is measurable if there is an embedding $i\colon V \to M$ such that κ is the least ordinal such that $i(\kappa) > \kappa$.

If i is such an embedding, let

$$U = \{\, x \subseteq \kappa : \kappa \in i(x) \,\}.$$

We call the set U a measure. It is

- (κ-complete) $X \subseteq U$ and $|X| < \kappa$ implies $\bigcup X \in U$,
- (uniform) $x \in U$ implies $|x| = \kappa$,
- (closed upward) $x \supseteq y \in U$ implies $x \in U$,
- (maximal) $x \subseteq \kappa$ implies $x \in U$ or $\kappa \setminus x \in U$,
- (Normal) if $f\colon \kappa \to \kappa$ and $\{\, \alpha < \kappa : f(\alpha) < \alpha \,\} \in U$ then there is γ so $\{\, \alpha : f(\alpha) = \gamma \,\} \in U$.

No measures is possible

Theorem (Scott 1961)

There are no measurable cardinals in L.

Theorem (Solovay 1960s)

Suppose that U is a measure on κ. Then in $L[U]$, κ is measurable and is the only measurable cardinal.

Theorem (Silver 1971)

The GCH holds in $L[U]$.

One measure is possible

Theorem (Kunen 1970)

- *If U is a measure in $L[U]$, then U is the only measure.*
- *More generally, suppose that U and U' are measures on κ and κ' in $L[U]$ and $L[U']$, respectively.*
  - *If $\kappa = \kappa'$ then $U = U'$.*
  - *If $\kappa < \kappa'$ then $L[U']$ is an iterated ultrapower of $L[U]$.*

Proof.

Suppose U and U' are measures on κ in $L[U]$ and $L[U']$, and let $\mathcal{C}$ the the club filter on κ^{++}. Then

$$L[U] \xrightarrow{i^U_{\kappa^{++}}} \mathrm{Ult}_{\kappa^{++}}(L[U], U) = L[i^U_{\kappa^{++}}(U)] = L[\mathcal{C}]$$

$$L[U'] \xrightarrow[i^{U'}_{\kappa^{++}}]{} \mathrm{Ult}_{\kappa^{++}}(L[U'], U') = L[i^{U'}_{\kappa^{++}}(U')] = L[\mathcal{C}]$$

where $i^U_{\kappa^{++}}(U) = \mathcal{C} = i^{U'}_{\kappa^{++}}(U')$.
If we set $\Gamma = \left\{ \alpha : i^U_{\kappa^{++}}(\alpha) = i^{U'}_{\kappa^{++}}(\alpha) = \alpha \right\}$ then

$$L[U] \cong \mathrm{Hull}^{L[\mathcal{C}]}(\Gamma \cup \{\mathcal{C}\}) \cong L[U'].$$

□

The maximum number of measures is possible

Theorem

[Kunen-Paris 1970] There is a generic extension $L[U, G]$ of $L[U]$ with $2^{(2^\kappa)}$ measures on κ.
In this model, $2^\kappa = \kappa^+$ and $2^{\kappa^+} = 2^{2^\kappa}$ can be arbitrarily large.

The Question

From the list of problems at the end of the Kunen-Paris paper:

> **Problem 1.** *Can the number of normal ultrafilters on a measurable cardinal be intermediate between 1 and 2^{2^κ}? Can this number be 2?*

From the chapter "Combinatorics" by Ken Kunen in the *Handbook of Mathematical Logic*, ed. by Jon Barwise, 1977.

> *Weakly compact cardinals are the beginning, not the end, of large cardinal theory. This game has been played to much greater heights (see* SILVER *[1973],* SHOENFIELD, *[1971],* SOLOVAY, REINHGART AND KANAMORI, *[∼1977]). It should be pointed out that it is cheating to name a number that does not exist. At times in the past, plausible large cardinal assumptions have turned out to be inconsistent with ZFC. But maybe ZFC is inconsistent?*

Appendix.* **Ridiculously large cardinals**.

. . .

* *Appended by the editor and set-theory coordinator.*

Where I came in

When finishing grad school, I noted the following observation of Kunen:

Proposition

Let A be the class of measurable cardinals. Then for each measurable cardinal κ there is a measure U_κ such that $A \cap \kappa \notin U_\kappa$.

Proof.

Suppose U_λ has been defined for $\lambda < \kappa$, and U is a measure on κ with $A \cap \kappa \in U$. Then set

$$U_\kappa = \{ x \subseteq \kappa : \{ \lambda \in A \cap \kappa : x \cap \lambda \in U_\lambda \} \in U \}.$$

□

Question

Set $\mathcal{U} = \langle U_\lambda : \lambda \leq \kappa \rangle ^\frown \langle U \rangle$. Are U_κ and U the only measures on κ in $L[\mathcal{U}]$?

The answer is yes

Definition

If U and U' are measures, say that $U \lhd U'$ if $U \in \mathrm{Ult}(V, U')$.

Let $o(U)$ be the $\lhd$-rank of U, and

$$o(\kappa) = \sup \{ o(U) : U \text{ is a measure on } \kappa \}.$$

Theorem

(Mitchell) Suppose that $o(\kappa) \geq \lambda$. Then there is an inner model $L[\mathcal{U}]$ of ZFC + GCH having either exactly $|\lambda|$ or $\kappa^{++} < \lambda$ many measures on κ.

- The proof is essentially the same as Kunen's proof that there is only one measure in $L[U]$.

Two further questions

- In the model $L[\mathcal{U}]$, there are many measurable cardinals below κ. Could we find a model in which κ has, say, exactly two measures, and κ is the only measurable cardinal?
- Is the large cardinal hypothesis that $o(\kappa) \geq \lambda$ necessary? Or could we start by assuming only that there is one measurable cardinal?

Theorem (Stu Baldwin 1983)

If $o(\kappa) \geq 2$ then there is a model in which κ is the only measurable cardinal, and κ has exactly two measures.

Sketch of Baldwin's proof

- First try: Use the model $M = L[U_\kappa, U, A]$ where, as before, $A = \{\, \lambda < \kappa : o(\lambda) \geq 1 \,\}$, and U_κ and U are measures on κ such that $A \in U - U_\kappa$.

  — Problem: We have no control over the set A, which might be moved differently by the two measures on κ.
- Use another technique from Kunen's thesis: Take iterated ultrapowers using ultrafilters which are not members of the model M (but are sufficiently close to M).
- The actual model has the form $M = L[U_\kappa, U, A, \mathcal{F}]$ where the set $\mathcal{F}$ is chosen so that the measures U_λ for $\lambda \in A$ are "sufficiently close" to M.

How to start with less

- When we started by assuming $o(\kappa) = 2$, then we already had 2 measures to start with.
- Starting with only one measure on κ, we need to add more by forcing.
- The main tool goes back to the Kunen-Paris result.

The Kunen-Paris construction

- Suppose that $i\colon M \to N$ is an embedding: say $N = \mathrm{Ult}(M, U)$ and $i = i^U$.
- Now suppose $G \subseteq P$ is M-generic, and we want to extend i to

$$i^*\colon M[G] \to N^*.$$

- Then we must have $N^* = N[G^*]$, where $G^* \subseteq i(P)$ is N-generic and $i[G] \subseteq G^*$, so

$$i^*\colon M[G] \to N[G^*] \text{ is defined by } i^*(\tau^G) = (i(\tau))^{G^*}.$$

- Furthermore, in order to have i^* definable in $M[G]$, we would want to have $G^* \in M[G]$.

The Kunen-Paris construction

- Kunen and Paris used Easton forcing to add subsets to add subsets of successor cardinals only, up to κ^+. Then $j(P) = P \times P^*$ where P^* is sufficiently closed that $N[G]$-generic subsets can be defined in $M[G]$—and so homogeneous that 2^{κ^+} of them can be.
- To end up with exactly two measures, the forcing $i(P)$ must be rather rigid — so that there are only two possible choices of G^* — but not entirely rigid, or there would be only one.

Theorem (Jeff Leaning 1999)

Suppose that $\langle U_\lambda : \lambda \leq \kappa \ \&\ \lambda$ *is measurable*$\rangle$ *is a sequence of measures such that*

$$\forall x \in U_\kappa \, \exists \lambda < \kappa \, (x \cap \lambda \in U_\lambda).$$

Then in a generic extension there is a model $L[B, U_0, U_1]$ *in which* κ *is the only measurable cardinal, and* U_0 *and* U_1 *are the only measures on* κ*.*

Sketch of proof

- The set B comes from a modified Prikry forcing which does not distinguish between the measurable cardinals.
- Conditions are pairs (b, D) in which
  - b is a finite set of ordinals less than κ
  - D is a set such that $D \cap \lambda \in U_\lambda$ for every measurable cardinal $\lambda < \kappa$.
- Then the measure U_1 is generated by the set $U \cup \{B\}$ and U_0 is generated by the set $U \cup \{\kappa - B\}$.

Something quite different

Theorem (Apter,Cummings,Hamkin 2007)

If κ is a measurable cardinal, then there is a forcing extension, neither creating nor destroying any measurable cardinals, where there are exactly κ^+ many normal measures on κ.

The proof uses an iteration of three generic extensions:

- The first uses the Kunen-Paris forcing to get 2^{2^κ} measures.
- The second adds a Cohen real.
- The third collapses 2^{2^κ} onto κ^+.

By a result of Hamkins (using the second stage) the third stage doesn't add any new measures. Thus in the final model there are exactly $(2^{2^\kappa})^{(V)}$ many measures, and this is κ^+ in the final model.

A final answer

Theorem (Friedman-Magidor 2008)

Assume that U is a measure on κ in $L[U]$. Then for any cardinal $\lambda \leq \kappa^{++}$, there is generic extension of $L[U]$, with the same cardinals, such that there are exactly λ measures on κ.

A sketch of the proof

We'll consider $\lambda = 2$.
Recall that we want a forcing order $P \in L[U]$ such that if $G \subseteq P$ is $L[U]$-generic then there are exactly two sets G^* such that

- $G^* \subseteq i(P)$,
- $i[G] \subseteq G^*$, and
- $G^* \in L[U][G]$.

Giving two embeddings $i^* \colon L[U][G] \to L[i(U)][G^*]$ in $L[U][G]$, and hence two measures.

Ignore, for the moment, the requirement that $G^* \in L[U][G]$.

- (S. Friedman and K. Thompson) — Use κ-Sacks forcing: conditions are binary trees of height κ in which all maximal branches have length κ and the set of branch nodes on any branch is closed and unbounded.
- Let b be a generic branch, and consider embeddings — not (yet) in $L[U]$

$$i^* \colon L[U][b] \to M[b^*] \qquad \text{with} \quad b^* = i^*(b).$$

Amazingly, there are exactly two choices of b^*:

- $b^*(\kappa)$ can be 0 or 1 — but that is the only possible variation.
- For any α with $\kappa < \alpha < i(\kappa)$ there is f such that $\alpha = [f]_U$.
- There is a dense set of conditions T which, for a club set of ordinals $\nu < \kappa$, have no spliting nodes in the interval $(\nu, f(\nu)]$. Such a condition (together with the choice of $b^*(\kappa)$) determines $b^* \upharpoonright (\kappa, \alpha]$.

- To get i^* in the generic extension, Friedman and Magidor use a backward Easton iteration culminating in the Sacks forcing at κ.
- To ensure that the image of the forcing below κ is unique they
  - for $\lambda < \kappa$ add, in addition to a Sacks set b_λ, a coding C_λ of b_λ using almost disjoint stationary subsets of λ^+, and
  - allow as supports in the iteration, sets $A \subseteq \kappa$ with $A \cap \lambda$ nonstationary for all inaccessible $\lambda \leq \kappa$.
- This ensures that the choice of $b_\kappa(\kappa) = 0$ or $b_\kappa(\kappa) = 1$ is the only possible variation in the choice of G^*. Thus there are exactly two measures.

□