Universality for linear orders, trees and lines

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Universality

Definition

Let $\mathscr{A}$ be a set of models (usually of the same theory) each of which has the same cardinality. A *universal model* for $\mathscr{A}$ is a model $U \in \mathscr{A}$ such that all models in $\mathscr{A}$ embed into U.

Embeddings

Ordered structures: an injective map that preserves order.

For partial orders, like trees, we may also require that incomparability is preserved, called *strong embedding*.

Linearly ordered topological space (LOTS) embeddings: injective continuous order-preserving map.

Classic model theory

Fact

(GCH) Any set of models of a first order theory of countable size has universal models in every uncountable cardinal.

For regular cardinals, this model is saturated.
For singular cardinals, it is special.

However, this does not apply to LOTS or any other theory which is not first-order definable.

Known results

- Under GCH, there are universal linear orders in all infinite cardinals. For regular cardinals κ, this model is $\mathbb{Q}(\kappa)$ which is the unique linear order of size κ without endpoints with the κ-*density property:*

$$\forall S, T \in [\mathbb{Q}(\kappa)]^{<\kappa}\ [S < T \Rightarrow (\exists x) S < x < T].$$

  Due to Hausdorff.
- (Shelah) $\neg$CH: The existence of a universal linear order at $\aleph_1$ is independent.
- (Kojman, Shelah) For $\lambda \in (\aleph_1, 2^{\aleph_0})$ regular, there is no universal linear order of size λ.

Successors of regulars

Let λ be an infinite regular cardinal.

Definition

We say that a forcing P has the λ-*covering property* iff whenever $p \in P$ and $\underset{\sim}{B}$ is a P-name for a subset of V such that $p \Vdash |\underset{\sim}{B}| = \lambda$, there is $q \leq p$ and $C \in V$ of size λ such that $q \Vdash \underset{\sim}{B} \subseteq C$.

Theorem

Let $V \vDash 2^{<\lambda} = \lambda$ and let P be a product forcing $\langle P_\alpha : \alpha < \lambda^{++} \rangle$ with the following properties:

- *δ-support for some $\delta \leq \lambda$,*
- *λ-covering property*
- *preserves λ^{++}*
- *for all α the P_α-generic is $A_\alpha \subset \lambda$.*

Then in V^P there is no universal linear order of size λ^+.

Global result

Corollary

There is a model of ZFC + not GCH such that at every regular cardinal λ, there is no universal linear order at λ^+ and at limit cardinals λ, there is a universal linear order at λ.

Model is built using an Easton product forcing λ^{++}-many Cohen subsets of λ for each regular λ.

Adding only λ^{++}-many subsets of λ assures that $\lambda^{<\lambda} = \lambda$ continues to hold at limit cardinals.

A better global result?

What's missing? Non-trivial result for successors of singular cardinals.

What's the problem? Need $2^\lambda > \lambda^+$ for singular λ, i.e. a failure of the SCH.

At successors of singulars

Definition

A cardinal κ is $(\kappa+\alpha)$-hypermeasurable iff there exists $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, M is closed under κ-sequences and $H(\kappa^{+\alpha})^V = H(\kappa^{+\alpha})^M$.

Theorem

Assume $M \models GCH$ and κ is is a $(\kappa+2)$-hypermeasurable cardinal. There is a forcing P such that in M^P there is no universal linear order of size κ^+, κ is singular of cofinality $\aleph_0$ and $2^\kappa > \kappa^+$.

Similar result proved for graphs, technique is joint work with S. Friedman.

Trees and universality

Well-orders are (very thin) trees.

There are no universal well-orders in any infinite cardinal.

For similar reasons there are no universal trees.

No assumptions apart from ZFC needed!

Omitting substructures

Theorem (T.)

Assume that $\lambda^{<\kappa} = \lambda$. Then there is no universal model in the following sets (representatives under isomorphism) of structures:

Linear orders of size λ which omit increasing κ-sequences

Linear orders of size λ which omit decreasing κ-sequences

Trees of size λ which omit κ-branches

Well-founded posets of size λ which omit κ-chains

Posets of size λ which omit κ-chains.

Linear orders results proved by Todorcevic and Väänänen for the case of $\lambda = \kappa = \aleph_1$.

Positive universality results

Theorem (T.)

Assuming GCH, let λ be a (strong) limit cardinal with $\lambda > \kappa \geq \aleph_0$ and cf$(\kappa) >$ cf(λ). Then there exists a (strong) universal in

- *Linear orders of size λ which omit decreasing κ-sequences*
- *Trees of size λ which omit κ-branches*
- *Well-founded posets of size λ which omit κ-chains.*

Universality spectra

Corollary

Assuming GCH, there is a universal model for

- *Linear orders of size λ which omit decreasing κ-sequences*
- *Trees of size λ which omit κ-branches and*
- *Well-ordered posets of size λ which omit κ-chains*

iff λ is a (strong) limit cardinal with $\lambda > \kappa \geq \aleph_0$ and cf$(\kappa) >$ cf(λ).

Other restricted classes of linear orders

- (Cantor) The reals $(\mathbb{R}, <)$ form a universal separable linear order.
- (Moore) PFA implies there exists a universal Aronszajn line.

LOTS vs. linear orders

If there is no universal linear order in a certain cardinality, then there is no universal LOTS in that cardinality.

In general, there exist injections which are order-preserving but not continuous. Also, there are injections which are continuous but not order-preserving.

Under GCH the classic result about the existence of uncountable universal models does not hold for LOTS.

HOWEVER ...

Countable universal LOTS

Let $(\mathbb{Q}, <, \tau)$ be the LOTS whose underlying linear order is the rationals and which is endowed with the open interval topology τ.

Claim

$(\mathbb{Q}, <, \tau)$ *forms a universal LOTS at* $\aleph_0$.

Regular uncountable LOTS

Assume that κ is a regular uncountable cardinal such that $\kappa = \kappa^{<\kappa}$.

$\bar{\mathbb{Q}}(\kappa)$ is the completion of $\mathbb{Q}(\kappa)$ under $< \kappa$-sequences.

Claim

Neither $(\mathbb{Q}(\kappa), <, \tau)$ nor $(\bar{\mathbb{Q}}(\kappa), <, \tau)$ are universal LOTS at κ.

For any infinite limit ordinal $\alpha < \kappa$, the linear orders $\alpha + 1$ and $\alpha + 1 + \alpha^*$, respectively, together with the open interval topology are counterexamples to universality.

Singular uncountable LOTS

At singular cardinals κ such that $\mathrm{cf}(\kappa) = \lambda < \kappa$, there is a special model $\mathscr{A}$ for linear orders, namely the union of the elementary chain of models $\mathbb{Q}(\aleph_{\alpha_i})$ for $\langle \alpha_i : i < \lambda \rangle$, a strictly increasing sequence of regular ordinals such that the $\aleph_{\alpha_i}$s are cofinal in κ.

The LOTS $\lambda^+ + 1$ also provides a counterexample to the LOTS universality of $\mathscr{A}$.

Uncountable universal LOTS: open questions

- Assuming GCH, for which uncountable cardinals do LOTS have a universal model?
- A universal LOTS for regular, uncountable cardinals must be also universal for linear orders, but cannot be the saturated order. Can there be a universal LOTS in a model where the only universal linear order is not saturated?