ALGEBRA QUALIFYING EXAM, AUGUST 2015

1. Let *groups*(n) denote the number of groups of order n up to isomorphism.

(a) Let p be a prime number. Show that every group of order p^2 is abelian, and determine *groups*(p^2).

(b) Find *groups*(50) (hint: show that every group of order 50 must have a normal Sylow 5-subgroup.)

2. Let k be a field of characteristic $\mathrm{char}(k) \neq 2$. Consider the $k[t]$-algebra

$$A = k[x,t]/(x^2 - t).$$

For every $a \in k$, let

$$A_a = A \otimes_{k[t]} k[t]/(t-a).$$

Do not assume that k is algebraically closed.

(a) Show that A is a flat $k[t]$-algebra.

(b) How many prime ideals are there in A_a? For each prime ideal $\mathfrak{p} \subset A_a$, describe the corresponding residue field (the field of fractions of $A_a/\mathfrak{p}$) as an extension of k. The answer may depend on a.

(c) For a certain value of a, the ring A_a has an ideal which is primary but not prime. What is the value a and what is the primary ideal?

3. Let $A \in \mathrm{GL}(n, \mathbb{C})$ be an $n \times n$ invertible matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $V = M_n(\mathbb{C})$ be the n^2-dimensional vector space of $n \times n$ matrices. Find the eigenvalues of the linear map

$$T : V \to V : M \mapsto A^{-1}MA$$

(conjugation by A). If you assume that the eigenvalues of A are pairwise distinct, you get partial credit.

4. (a) Let A be a commutative subalgebra of $M_n(\mathbb{C})$, the algebra of $n \times n$ matrices over the complex numbers. Suppose that A contains $\mathbb{C}$ (thought of as the subalgebra of scalar matrices) and that it is generated as a $\mathbb{C}$-algebra by a single element. Show that $\dim_{\mathbb{C}}(A) \leq n$.

(b) Let $B \subset M_n(\mathbb{C})$ be the set of matrices that can be expressed as $\lambda I_n + N$, where $\lambda \in C$ is a scalar and N is strictly upper-triangular. Thus, elements of B are upper-triangular matrices with the same scalar λ on the diagonal. Show that B is a subalgebra.

(c) When $n = 4$, give an example of a commutative subalgebra of B_4 (and thus of M_4) of dimension 5.

5. Let $\mathbb{C}(x)$ be the field of rational functions with complex coefficients of the variable x. Thus, x is transcendental over $\mathbb{C}$. Put

$$y = x^n + x^{-n} \in \mathbb{C}(x)$$

for some $n > 0$.

Prove that the field extension $\mathbb{C}(x)/\mathbb{C}(y)$ is a finite Galois extension and find its degree.