# ALGEBRA QUALIFYING EXAM, AUGUST 2016

**1.** Let $G$ be a group. By a *maximal subgroup* of $G$ we mean a subgroup $M \neq G$ such that the only subgroups containing $M$ are $M$ and $G$.

(a) Describe all the maximal subgroups of the dihedral group of order $2p$, where $p$ is an odd prime. How many are there?

(b) Show that if a finite group $G$ has only one maximal subgroup, then $G$ is cyclic.

(c) Show that if a maximal subgroup $M \subset G$ is normal, then the index of $M$ in $G$ is finite and prime.

**2.** Let $V$ denote a nonzero finite-dimensional vector space over the complex field $\mathbb{C}$. Given a linear transformation $A : V \to V$, show that the following are equivalent:

(i) There exists a linear transformation $P : V \to V$ such that $P^2 = I$ and $AP = -PA$;

(ii) There exists an invertible linear transformation $P : V \to V$ such that $AP = -PA$;

(iii) There exists a direct sum decomposition $V = V_1 \oplus V_2$ such that $AV_1 \subseteq V_2$ and $AV_2 \subseteq V_1$.

**3.** Let $R$ be the subring of $\mathbb{C}[x, y]$ consisting of the polynomials $P(x, y)$ such that $P(x, y) = P(y, x)$.

(a) Show that $R$ is generated as a $\mathbb{C}$-algebra by $x + y$ and $xy$.

(b) Show that the map $\mathbb{C}[u, v] \to R$ sending $u$ to $x + y$ and $v$ to $xy$ is an isomorphism.

(c) Let $S$ be the subring of $\mathbb{C}[x, y]$ consisting of the polynomials $P(x, y)$ such that $P(x, y) = P(-x, -y)$. Can $S$ be generated as a $\mathbb{C}$-algebra by two polynomials? Either give two generators, or prove that $S$ is not generated by 2 polynomials.

**4.** Let $R$ be a commutative ring. A prime ideal $\mathfrak{p} \subset R$ is called *minimal* if $\mathfrak{q} \subseteq \mathfrak{p}$ for a prime ideal $\mathfrak{q}$ implies that $\mathfrak{q} = \mathfrak{p}$.

(a) Determine the minimal primes of $R = k[x, y]/(xy)$, where $k$ is a field.

(b) Prove that if there exists a surjective map of $R$-modules $R^m \to R^n$ for positive integers $m$ and $n$ then $m \geq n$.

(c) Assume that $R$ has no nilpotents. Prove that if there exists an injective map of $R$-modules $R^m \to R^n$ then $m \leq n$. Hint: show that under these assumptions, $R_{\mathfrak{p}}$ is a field when $\mathfrak{p}$ is minimal. You may use without proof that minimal primes exist in any commutative ring.

**5.** Put $\alpha := e^{\frac{2\pi\sqrt{-1}}{7}}$, and consider the field $K := \mathbb{Q}(\alpha)$. Find an element $x \in K$ such that $[\mathbb{Q}(x) : \mathbb{Q}] = 2$. (Proving that such $x$ exists would earn you partial credit; for full credit, express $x$ explicitly as a polynomial in $\alpha$, such as $42\alpha^3 - 1337\alpha^5$.)