Putnam Club 2018
Techniques from Group Theory

Sets Closed Under Binary Operations

Definition: Let S be a set. A *binary operation* on S is a function $S^2 \to S$. We sometimes say S is *closed* under the operation.

Useful concepts:

- Associativity, commutativity, identities, (one sided) inverses are terms you should be familiar with.
- If identities or inverses exist, then they are unique!
- A "(left) cancellation" rule is one in which the following holds:

$$x \cdot y = x \cdot z \Rightarrow y = z.$$

  A similar statement holds for right cancellation.

- Technique: clever substitutions!

1. Let S be a set with binary operation $\cdot$. Suppose $(a \cdot b) \cdot a = b$ for all $a, b \in S$. Show $a \cdot (b \cdot a) = b$ for all $a, b \in S$.
2. Let S be a finite set with at least four elements and an associative binary operation. Suppose

$$(a \cdot a) \cdot b = b \cdot (a \cdot a) = b$$

   for all $a, b \in S$. Show that

$$S = \{a \cdot (b \cdot c) | a, b, c \in S, a \neq b, b \neq c, a \neq c\}.$$

3. Let S be the smallest set of rational functions containing $f(x, y) = x$ and $g(x, y) = y$ which is closed under subtraction and taking reciprocals. Show that the only constant function in S is the zero function.
4. Let S be a set with an associative binary operation so that

$$a \cdot b = b \cdot a \Rightarrow a = b.$$

   Show that $a \cdot (b \cdot c) = a \cdot c$ for all $a, b, c \in S$.

Groups

Definition: A *group* is a set G and a binary operation $(\cdot)$ so that G is closed under $\cdot$ and

- $\cdot$ is associative.
- There is an identity element $e \in G$.
- All elements in G have inverses.

Useful concepts:

- Abelian $\leftrightarrow$ commutative.
- Be familiar with standard groups: permutation groups S_n, cyclic groups Z_n, symmetry groups S_{2n}, Klein group K_4, the infinite groups of integers, rationals or reals, matrix groups.
- Kronecker: A nontrivial subgroup of the real numbers under addition is either cyclic or it is a dense subset.
- The *order* of an element a of a group is the smallest positive integer n so that $a^n = a \cdot a \cdots a = e$.

1. Let G be a set closed under an associative operation $\cdot$. Suppose $(G, \cdot)$ has a left cancellation rule and that there is $a \in G$ so that $x^3 = axa$ for all $x \in G$. Prove that $(G, \cdot)$ is an Abelian group.

2. Let $M = \mathbb{R} - \{3\}$. Let $m \in \mathbb{R}$ and define

$$x \cdot y = 3(xy - 3x - 3y) + m.$$

For what values of m will $(M, \cdot)$ be a group?

3. Let r, s, t be relatively prime positive natural numbers. Let a, b be elements of an Abelian group with identity e. Suppose $a^r = b^s = (ab)^t = e$. Then $a = b$.

4. Let G be a finite group and $A \subset G$. If A has more than half of the elements of G then every element of G can be expressed as a product of elements in A.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and

$$f(x) + f(x + \sqrt{2}) = f(x + \sqrt{3})$$

for all $x \in \mathbb{R}$. Show f is constant.

6. Prove that the sequence $a_n = \sin n$ is dense in $[-1, 1]$.

7. Prove that the group of invertible matrices in $\mathbb{R}^{4\times 4}$ with rational entries has no element of order seven.

8. Let G be a finite group of invertible matrices with complex entries. Define M as the sum of the matrices in G. Show that the determinant and trace of M are both integers.