ALGEBRA QUALIFYING EXAM, JANUARY 2014

In all cases, when an example is requested, you should both provide the example and a proof that the object you write down actually is an example.

(1) If R is a commutative ring and I an ideal, we denote by $\sqrt{I}$ the *radical* of I, which is by definition the set of $r \in R$ such that $r^n \in I$ for some natural number n.
  (a) Prove that $\sqrt{I}$ is an ideal of R.
  (b) Give an example of an ideal I in $\mathbb{Q}[x, y]$ such that I is non-principal but $\sqrt{I}$ is principal.
  (c) This notion of radical doesn't work so well for noncommutative rings, even for two-sided ideals. For instance, we might define $\sqrt{0}$ in the ring $M_2(\mathbb{R})$ of 2×2 matrices to be the set of all elements r such that $r^n = 0$ for some natural number n. Show that, with this definition, $\sqrt{0}$ is *not* an ideal.

(2) Let F be a field and n a positive integer. Let A be an n by n matrix over F, such that A^n is zero but A^{n-1} is nonzero. Show that any n by n matrix B over F that commutes with A is contained in the F-linear span of $I, A, A^2, ..., A^{n-1}$.

(3) Let G be a finite group.
  (a) If H is a proper subgroup of G, show that there is some element $x \in G$ which is not contained in any subgroup conjugate to H.
  (b) A *maximal subgroup* of G is a proper subgroup which is not contained in any other proper subgroup. Derive from the first part of the problem that if all maximal subgroups of G are conjugate, G must be cyclic.

(4) Recall that a module M for a commutative ring R is called *torsion* if, for each m in M, there exists a nonzero element of r such that $rm = 0$, and M is called *torsion-free* if $rm = 0$ implies that $r = 0$ or $m = 0$.
  (a) If M and N are torsion modules, prove that $M \otimes_R N$ is torsion.
  If M and N are torsion-free modules, on the other hand, $M \otimes_R N$ need not be torsion-free. For example, let I be the ideal (x, y) in the ring $R = \mathbb{C}[x, y]$, which is torsion-free because R is an integral domain. Prove that $I \otimes_R I$ is not torsion-free, by means of the following two steps:
  (b) Show that $x \otimes y - y \otimes x \in I \otimes_R I$ is a torsion element;
  (c) Show that $x \otimes y - y \otimes x \neq 0$.

(5) If L/K is a field extension, we say a field K' is *intermediate* between L and K if it properly contains K and is properly contained in L.
  (a) Prove that a field extension L/K of degree 4 has at most 3 intermediate fields. Give an example of a field extension L/K of degree 4 which has exactly 3 intermediate fields.
  (b) Give an example of a field extension L/K of degree 4 such that there is no intermediate field K' between L and K.