ALGEBRA QUALIFYING EXAM, JANUARY 2018

1. For this problem and this problem only your answer will be graded on correctness alone, and no justification is necessary.

(a) Give an example of a commutative ring R and a non-zero element $f \in R$ where the localization $R_f = 0$.

(b) Give an example of a commutative ring R and an element $f \in R$ where the localization map $R \to R_f$ is neither injective nor surjective.

(c) Give an example of a local ring R and an element $f \in R$ where $R_f \neq 0$, but R_f is no longer a local ring.

2. Recall that a (left) zero divisor in a ring R is an element a such that $ab = 0$ for some nonzero $b \in R$. Consider the rings

$$R_1 = \mathbb{C}[x]/(x^3) \qquad \text{and} \qquad R_2 = M_n(\mathbb{C}) \quad (n \times n \text{ matrices over } \mathbb{C}, \text{ where } n > 1).$$

(a) Give an example of a nonzero zero-divisor in the ring R_1.

(b) Give an example of a nonzero left zero-divisor in the ring R_2.

(c) Prove that the set of zero-divisors of R_1 is an ideal, but the set of left zero-divisors of R_2 is not a left ideal.

(d) Let R be a commutative ring. Prove that if the set of zero-divisors in R is an ideal I, then $I \subset R$ is a prime ideal.

3. Consider the field F with 11 elements. Let G denote the cyclic group of order 11, with generator $r \in G$. Denote by FG the group algebra of G (also sometimes denoted by $F[G]$). We consider r as an element of FG, and let $T : FG \to FG$ be the F-linear map such that $T(x) = rx$ for all $x \in FG$. Find the Jordan canonical form of T.

4. Let G be a finite group. Denote by $\mathrm{Aut}(G)$ the group of automorphisms of G and by $Z(G) \subset G$ the center of G.

(a) Show that the quotient $G/Z(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(G)$.

(b) Show that if $G/Z(G)$ is cyclic, then G is abelian.

(c) Suppose that $\mathrm{Aut}(G)$ is a cyclic group. Show that G is abelian.

(d) Show that if G is abelian, then the map $\phi : x \mapsto x^{-1}$ is an automorphism of G.

(e) Deduce that there exists no group G such that $\mathrm{Aut}(G)$ is a nontrivial cyclic group of odd order (and, in particular, that $\mathrm{Aut}(G)$ is finite).

5. Let K be the splitting field of the polynomial $x^4 - x^2 - 1$ over $\mathbb{Q}$. Compute the Galois group of the extension $K/\mathbb{Q}$. (For partial credit, find the degree $[K : \mathbb{Q}]$.)

Solutions

1. (a) f is any nilpotent in R, for instance, $R = k[x]/(x^2)$ and $f = x$.
(b) For instance, $R = k[x] \times k[x]$ and $f = (0, x)$.
(c) For instance, $R = k[[x, y]]/(xy)$ and $f = x + y$.

2. (a) For instance, take x.
(b) Any non-invertible matrix.
(c) Suppose $f = a_0 + a_1x + a_2x^2 \in R_1$. If a_0 is nonzero, it is easy to construct an inverse to f, which means that f is not a zero-divisor. Thus any zero-divisor lies in (x), and conversely it is easy to see that every element $f \in (x)$ satisfies $fx^2 = 0$, and therefore is a zero-divisor. Thus the set of zero-divisors is the ideal (x).

By contrast, any singular matrix in R_2 is a zero-divisor. In particular, let M be the diagonal matrix $\mathrm{diag}(0, 1, \ldots, 1)$ and M' the diagonal matrix $\mathrm{diag}(1, 0, \ldots, 0)$. Then M and M are zero-divisors, but $M + M'$ is the identity, which is not a zero-divisor; so the set of zero-divisors is not an ideal.

(d) Suppose $a, b \in R$ are such that $ab \in I$. Then there exists some c such that $(ab)c = 0$. Then either $bc = 0$, in which case b is a zero-divisor, or $bc \neq 0$, in which case $a(bc) = 0$ implies that a is a zero-divisor, as claimed.

3. Write $p = 11$. The minimal polynomial of T has degree p, since for any nonzero polynomial f in $F[x]$ with degree less than p, the F-linear map $f(T)$ is nonzero:

$$f(T)(1) = f(r) \neq 0 \in FG.$$

By construction $T^p = I$, so $x^p - 1$ is both the minimal and characteristic polynomial of T. The field F has characteristic p so

$$x^p - 1 = (x - 1)^p.$$

So T has one eigenvalue 1 with algebraic multiplicity p and geometric multiplicity 1. By these comments the Jordan canonical form of T is a single $p \times p$ block with eigenvalue 1.

4. (a) For any $g \in G$, the conjugation by g is an automorphism of G; this defines a homomorphism $G \to \mathrm{Aut}(G)$. By definition, $Z(G)$ is the kernel, therefore by the isomorphism theorem $G/Z(G)$ is identified with the image, which is a subgroup of $\mathrm{Aut}(G)$ (consisting of inner automorphisms).

(b) Say $G/Z(G) = \langle g \rangle$. Then any element of G can be written as ug^n for $u \in Z(G)$; we now easily see that

$$(ug^m)(vg^n) = uvg^{(m+n)} = (vg^n)(ug^m) \qquad (u, v \in Z(G)).$$

(c) By (a), $G/Z(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(G)$. If $\mathrm{Aut}(G)$ is cyclic, so is $G/Z(G)$; now (c) follows from (b).

(d) This is an easy direct calculation.

(e) By (c), G is abelian. Since $\phi^2 = 1$ and $\phi \in \mathrm{Aut}(G)$, $\phi = 1$, so G is elementary 2-abelian. If $|G| = 2$, then $\mathrm{Aut}(G) = e$, otherwise $\mathrm{Aut}(G)$ is non-abelian.

5. Put $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, so that α and β are the roots of the polynomial $x^2 - x - 1$. The roots of $x^4 - x^2 - 1$ are then $\pm\sqrt{\alpha}$, $\pm\sqrt{\beta}$. The Galois group acts on the four roots by transposition. The action has the following properties: it includes a transposition (namely, complex conjugation), it preserves the partition $\{\pm\sqrt{\alpha}\}$, $\{\pm\sqrt{\beta}\}$ (because every automorphism sends α to itself or to β) and it includes an element that sends $\pm\sqrt{\alpha}$ to $\pm\sqrt{\beta}$ (because α and β are conjugates). This implies that the Galois group is the eight-element dihedral group.