Math 763. Homework 4
Due Thursday, October 17th

In these problems (and everywhere else in the class), the ground field, which is denoted by k, is assumed to be algebraically closed.

1. Consider the curve $X = V(x^3 + x^2 - y^2) \subset \mathbb{A}^2$. Show that the function $f = y/x$ is a birational isomorphism between X and $\mathbb{A}^1$. Find the domain of f and the domain of the map f^{-1}. Find non-empty open subsets in X and $\mathbb{A}^1$ such that f provides an isomorphism of these sets.

2. Let $X \subset \mathbb{A}^n$ be a hypersurface given by the equation

$$f_{m-1}(x_1, \dots, x_n) + f_m(x_1, \dots, x_n) = 0,$$

where f_{m-1} and f_m are non-zero homogeneous polynomials of degrees $m - 1$ and m, respectively. Prove that if X is irreducible, it is rational. (Shafarevich, Problem I.3.5.)

3. Prove that an irreducible quadric (that is, given by a degree two equation) hypersurface is rational.

4. Fiber product of varieties Let X, Y, Z be three varieties, and let $f : X \to Z$ and $g : Y \to Z$ be regular maps. The *fiber product* (or Cartesian product) $S := X \times_Z Y$ is a variety together with maps $\phi : S \to X$ and $\psi : S \to Y$ such that $f \circ \phi = g \circ \psi$ and the triple (S, ϕ, ψ) is universal (for any other such triple (S', ϕ', ψ'), there is a unique map $S' \to S$ making the natural diagram commute).

Show that the fiber product exists. Prove that it is always a locally closed subvariety of $X \times Y$. What condition would imply that it is closed?

(Remark: if $X \hookrightarrow Z$ is an embedding of a locally closed subvariety, then the fiber product is the preimage $g^{-1}(X)$.)

5. Let $f : X \to Y$ be a morphism of varieties. For every point $x \in X$, f induces a morphism of local rings

$$f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}.$$

Prove that f is an isomorphism if and only if it is a homeomorphism and f_x is an isomorphism for all $x \in X$.

6. Suppose X is separated. Prove that for any two affine open subset $U, V \subset X$, the intersection $U \cap V$ is affine. (Hint: consider $U \times V \subset X^2$.)

7. Let $X \subset \mathbb{A}^n$ be an irreducible hypersurface. Consider the projection $\pi : X \to \mathbb{A}^{n-1}$ onto one of the coordinate hyperplanes. Suppose that π is dominant.

Show that the induced map $k(\mathbb{A}^{n-1}) \hookrightarrow k(X)$ realizes $k(X)$ as a finite extension of $k(\mathbb{A}^{n-1})$ (this is almost proved in class). Put $\deg(\pi) := [k(X) : k(\mathbb{A}^{n-1})]$. Now show that there is a non-empty open subset $U \subset \mathbb{A}^n$ such that every $x \in U$ has exactly $\deg(\pi)$ preimages.