# Math 764. Homework 6

Due Friday, March 10th

*Sheaves of modules on ringed spaces.*

Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *tensor product* of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

**1.** Prove that the stalks of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ are given by the tensor product:

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x,$$

where $x \in X$. Conclude that the tensor product is a right exact functor (in each of the two arguments).

**2.** Suppose that $\mathcal{F}$ is locally free of finite rank. (That is to say, every point $x \in X$ has a neighborhood $U$ such that $\mathcal{F}|_U \simeq (\mathcal{O}_U)^n$. Prove that there exists a natural isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathcal{G} \otimes \mathcal{F}^\vee.$$

Here $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is the dual of the locally free sheaf $\mathcal{F}$, and $\mathcal{H}om$ is the sheaf of homomorphisms. (Note that $\mathcal{G}$ is not assumed to be quasi-coherent.)

**3.** (Projection formula) Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Suppose $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{G}$ is a locally free $\mathcal{O}_Y$-module of finite rank. Construct a natural isomorphism

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \simeq f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G}.$$

*Coherent sheaves on a noetherian scheme*

**4.** Let $\mathcal{F}$ be a coherent sheaf on a locally noetherian scheme $X$.

(a) Show that $\mathcal{F}$ is locally free if and only if its stalks $\mathcal{F}_x$ are free $\mathcal{O}_{X,x}$-modules for all $x \in X$.

(b) Show that $\mathcal{F}$ is locally free of rank one if and only if it is *invertible*: there exists a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \simeq \mathcal{O}_X$.

**5.** As in the previous problem, supposed $\mathcal{F}$ be a coherent sheaf on a locally noetherian scheme $X$. The *fiber* of $\mathcal{F}$ at a point $x \in X$ is the $k(x)$-vector space $i^*\mathcal{F}$ for the natural map $i : Spec(k(x)) \to X$ (where $k(x)$ is the residue field of $x \in X$). Denote by $\phi(x)$ the dimension $\dim_{k(x)} i^*\mathcal{F}$.

(a) Show that the function $\phi(x)$ is upper semi-continuous: for every $n$, the set $\{x \in X : \phi(x) \geq n\}$ is closed.

(b) Suppose $X$ is reduced. Show that $\mathcal{F}$ is locally free if and only if $\phi(x)$ is constant on each connected component of $X$. (Do you see why we impose the assumption that $X$ is reduced here?)

**6.** Let $X$ be a locally noetherian scheme and let $U \subset X$ be an open subset. Show that any coherent sheaf $\mathcal{F}$ on $U$ can be extended to a coherent sheaf on $\overline{\mathcal{F}}$ on $X$. (We say that $\overline{\mathcal{F}}$ is an *extension* of $\mathcal{F}$ if $\overline{\mathcal{F}}|_U \simeq \mathcal{F}$.)

(If you need a hint for this problem, look at Problem II.5.15 in Hartshorne.)