Math 764. Homework 9
Due Wednesday, April 22nd

Coherent sheaves on $\mathbb{A}^1$

1. A coherent sheaf on an irreducible variety X is said to be *torsion* if its fiber at any one point is zero; semicontinuity then implies that the sheaf is zero on a non-empty open subset of X.

Classify coherent torsion sheaves on $\mathbb{A}^1$ up to isomorphism.

2. Show that any vector bundle on $\mathbb{A}^1$ is trivial.

Tensor product.

Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. Their *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ (over $\mathcal{O}_X$) is defined to be the sheafification of the pre-sheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

By construction, it is a sheaf of $\mathcal{O}_X$-modules.

3. Prove that the stalks of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ are given by tensor product:

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x.$$

4. Suppose that $\mathcal{F}$ is locally free of finite rank. Prove that there exists a natural isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathcal{G} \otimes \mathcal{F}^\vee.$$

Here $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is the dual of the locally free sheaf $\mathcal{F}$, and $\mathcal{H}om$ is the sheaf of homomorphisms.

5. Suppose now that X is a variety. Prove that tensor product of quasi-coherent sheaves is quasi-coherent.

6. Show that tensor product commutes with fibers: if $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent sheaves on a variety X, then the fiber of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ at any point $x \in X$ is the tensor product of the fibers of $\mathcal{F}$ and $\mathcal{G}$.

7. A locally free sheaf of rank one is sometimes called an *invertible sheaf*. Show that this name is accurate: if $\mathcal{F}$ is a coherent sheaf and there exists $\mathcal{G}$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \simeq \mathcal{O}_X$ ('invertibility') then $\mathcal{F}$ is locally free of rank one. (The question becomes trickier if we consider the same condition, but $\mathcal{F}$ is only assumed to be quasi-coherent.)