MATH 240

Midterm #1 · Section 3

OCTOBER 10, 2013

NAME:

GRADE

Instructions:

1. This Midterm consists of six questions. The total points for each of them collected in the table below.
2. Each question must be answered clearly on a **separate sheet of paper**, **ink** and **detail any reasoning used to justify**.
3. **No** notes, books, pagers, cell phones or electronic devices are **allowed**.
4. The duration of this test is **1 hour** and **15 minutes**.

QUESTION	*POINTS*	*SCORE*
1	*15*	
2	*20*	
3	*15*	
4	*15*	
5	*15*	
6	*20*	
	TOTAL	

1. Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent. [*15 points*]

Solution: Let us construct the truth table for the given propositions. For proposition $p \leftrightarrow q$ we have:

Table 1: Truth Table for $p \leftrightarrow q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	F
F	F	T

While for $(p \wedge q) \vee (\neg p \wedge \neg q)$ we get:

Table 2: Truth Table for $(p \wedge q) \vee (\neg p \wedge \neg q)$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
T	T	F	F	T	F	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	F	T	T	F	T	T

Since they have the same final column in their respective truth tables we conclude that both propositions are logically equivalents.

2. Construct the truth table for the compound propositions

   (a) $\big[(p \to q) \wedge (\neg p \to r)\big] \to q \vee r.$ [*10 points*]

   (b) $\big[(p \leftrightarrow q) \wedge (r \to q)\big] \to (r \to p).$ [*10 points*]

Solution:

(a) For the first proposition we have

Table 3: Truth Table for $\big[(p \to q) \wedge (\neg p \to r)\big] \to q \vee r$

p	q	r	$\neg p$	$p \to q$	$\neg p \to r$	$(p \to q) \wedge (\neg p \to r)$	$q \vee r$	$\big[(p \to q) \wedge (\neg p \to r)\big] \to q \vee r$
T	T	T	F	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	F	T	F	T	T
T	F	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	F	F	F	T

(b) And for the second

Table 4: Truth Table for $\big[(p \leftrightarrow q) \wedge (r \to q)\big] \to (r \to p)$

p	q	r	$p \leftrightarrow q$	$r \to q$	$r \to p$	$(p \leftrightarrow q) \wedge (r \to q)$	$\big[(p \leftrightarrow q) \wedge (r \to q)\big] \to (r \to p)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	T
T	F	F	F	T	T	F	T
F	T	T	F	T	F	F	T
F	T	F	F	T	T	F	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

3. Recall that $\lfloor x \rfloor = \max \{ k \in \mathbb{Z} \mid k \leq x \}$. If $f : \mathbb{R} \to \mathbb{Z}$ is given by $f(x) := \lfloor x \rfloor$ then (justify your answer!)

   (a) f is injective, i.e., one-to-one.

   (b) f is surjective, i.e., onto. [*15 points*]

   (c) f is bijective, i.e., one-to-one and onto.

Solution:

(a) Observe that for any integer k, if $k \leq x < k + 1$ then $\lfloor x \rfloor = k$. Therefore, f is many to one and hence not injective.

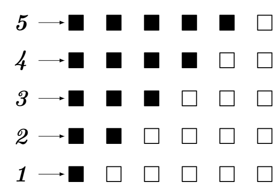
(b) Since $f(k) = k$ for all integer k, f is onto.

(c) A function is bijective if, and only if, it is one-to-one and onto. Hence f is not bijective.

4. (a) Find the sum $1+2+3+\cdots+100$ (No calculator. Explain!). [*8 points*]

   (b) Use the ideas to obtained (a) and calculate the sum $1+3+5+\cdots+99$. [*7 points*]

Hint: Work out (a) by analyzing (then generalizing) the following figure. . .



Solution:

(a) A shown in the given figure, if we construct a rectangle grid with 100 rows and 101 columns of black/white rectangles, half of them are black and will account for the desired sum. Since the total number of black/white rectangles is $101 \cdot 100$, we conclude that

$$1+2+3+\cdots+100 = \frac{101 \cdot 100}{2} = 101 \cdot 50 = 5050.$$

(b) If $s\colon = 1+3+5+\cdots+99$ and $t\colon = 2+4+6+\cdots+100$ then, clearly, $s+t = 1+2+3+\cdots+100 = 5050$ (by (a)). Now we observe that, arguing as in (a), $t = 2(1+2+3+\cdots+50) = 50 \cdot 51 = 2550$. Therefore $s = 5050 - 2550 = 2500$.

5. Does there exist any 2×2 matrix A such that

$$A \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot A?$$

If so,

(a) is it unique, [*15 points*]

or

(b) there are infinitely many of them.

Solution: It we put

$$A\colon = \begin{pmatrix} x & y \\ z & t \end{pmatrix},$$

then

$$A \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 2x+y \\ z & 2z+t \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x+2z & y+2t \\ z & t \end{pmatrix}.$$

Imposing the giving condition on A we have

$$\begin{cases} x &= x+2z \\ 2x+y &= y+2t \\ 2z+t &= t \end{cases} \quad \Rightarrow \quad \begin{cases} z &= 0 \\ x &= t \\ z &= 0. \end{cases}$$

This means that the solutions matrices are given by

$$\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$$

for arbitrary $x, y \in \mathbb{R}$. Hence there are always infinitely many solutions and, therefore, (b) holds and so (a) does not.

Remark. To resolve this question one could argue right away that, since the identity commutes with any matrix, every multiple $A = xI$ $(x \in \mathbb{R})$ of it will, obviously, be a solution (observe that this is just the case $y = 0$ above).

6. Let $m > 1$ be an integer. Recall that a congruence class $[a]_m \in \mathbb{Z}_m - \{[0]_m\}$ is called a zero divisor if there exists another class $[b]_m \in \mathbb{Z}_m - \{[0]_m\}$ such that $[a]_m \cdot [b]_m = [0]_m$.

   (a) Show that zero divisors do not have multiplicative inverses[*]. [*10 points*]

   (b) Show that if a and m have a common divisor $d > 1$ then $[a]_m$ is a zero divisor. [*10 points*]

Solution:

(a) If $[a]_m$ has a multiplicative inverse and $[b]_m \in \mathbb{Z}_m$ is such that $[a]_m \cdot [b]_m = 0$, then $[b]_m = 0$. This follows since if for some $[\kappa] \in \mathbb{Z}_m$, $[\kappa]_m \cdot [a]_m = [1]_m$, then

$$[b]_m = ([\kappa]_m \cdot [a]_m) \cdot [b]_m = [\kappa]_m \cdot ([a]_m \cdot [b]_m) = [\kappa]_m \cdot [0]_m = [0]_m.$$

Therefore, by the very definition, no zero divisor $[a]_m$ admits a multiplicative inverse.

(b) If $d > 1$ is a common divisor of a and m then $a = d \cdot \kappa$ and $m = d \cdot \ell$ for some integers κ and ℓ with $\ell < m$. Hence

$$a \cdot \ell = m \cdot \kappa \quad \Rightarrow \quad [a]_m \cdot [\ell]_m = [a \cdot \ell]_m = [0]_m,$$

and $[a]_m$ is a zero devisor unless $[a]_m = 0$, i.e., unless $m \mid a$[♮].

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[*] $[\kappa]_m$ is a multiplicative inverse of $[\ell]_m$ if $[\kappa]_m \cdot [\ell]_m = [\ell]_m \cdot [\kappa]_m = [1]_m$.

[♮] This conclusion was not originally stated since it is tacitly assume that a is a reminder modulo m and so $0 \leq a < m$.