NOTES FOR MATH 763: INTRODUCTION TO ALGEBRAIC GEOMETRY FALL 2012

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1. Sept. 4

Crowning achievement: Riemann-Roch theorem; basic theort of affine and projective algebraic varieties. Not much talk about schemes.

Approach with varieties is followed by Shafarevich (*Basic Algebraic Geometry*), also Milne's notes.

1.1. What is Algebraic Geometry?

Different geometries are roughly distinguished by the types of functions used to study them. Algebraic Geometry deals with the geometry of polynomial functions. Good news: polynomials are simple; bad news: there are very few polynomial functions (e.g. solving differential equations doesn't work, though AG has passive aggressive way of dealing with it: D-modules studies them without solving them). So Algebraic Geometry is more rigid: no nice deformations (again not enough functions).

Example. Consider graph of $x^2 + y^2 = 1$. This has a parametrization

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

which is not allowed. However, project (x, y) onto the line $(0, u)$ through the east point $(-1, 0)$ and it looks like:

$$\begin{cases} x = \dfrac{1-u^2}{1+u^2} \\ y = \dfrac{2u}{1+u^2} \end{cases}$$

.

Claim: algebraic parametriation has nice properties the arclength parametrization does not.

(1) Find all $(x, y) \in \mathbb{Q}^2$ such that $x^2 + y^2 = 1$. Answer: if and only if $u \in \mathbb{Q}$.
(2) How many solutions does $x^2 + y^2 = 1$ have in $\mathbb{Z}/43\mathbb{Z}$? Answer is $43 + 1$, obtained by $\left\{\left(\dfrac{1-u^2}{1+u^2}, \dfrac{2u}{1+u^2}\right) : u \in \mathbb{Z}/43\mathbb{Z}\}\right\}$, plus the point $\{(-1, 0)\}$.

    Exercise. How many solutions in $\mathbb{Z}/41\mathbb{Z}$? The answer is not $1 + 41$.
(3) Consider the integral

$$\int_{x^2+y^2=1} P(x, y)dx + Q(x, y)dy$$

where P and Q are rational.

Arclength parametrization gives $\int R(\sin(t), \cos(t))dt$ which is not nice.

Rational parametrization gives $\int \tilde{R}(u)du$ where $\tilde{R}$ is a rational function.

There are two ways to teach this course: through analysis, which restricts to filed of real numbers, and algebraic, which we will take. Our *main tool* will be (commutative) algebra, which works over any field.

What will we study? What we study changes.

(1) Algebraic sets
(2) Algebraic varieties
(3) Algebraic schemes
(4) Algebraic stacks
(5) $\vdots$

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1.2. **Algebraic sets.** Fix the ground field $\Bbbk$. Assume also that $\Bbbk = \bar{\Bbbk}$ (algebraically closed).

Definition. n-dimensional *affine space* is $\Bbbk^n$. *Polynomial functions* are $f \in \Bbbk[x_1, \dots, x_n]$. An *algebraic subset* $X \subset \Bbbk^n$ is the common zero locus of a set $S \subset \Bbbk[x_1, \dots, x_n]$.

Example. Plane curves: $x^2 + y^2 - 1 = 0$, $xy - 1 = 0$ (both smooth), $y^2 - x^3 = 0$ (singular), Non-planar twisted cubic $z - x^3 = 0$, $y - x^2 = 0$. Also (smooth) surface $z - (x^2 - y^2) = 0$.

Notation. Given $S \subset \Bbbk[x_1, \dots, x_n]$, define $V(S) = \{x \in \Bbbk^n : f(x) = 0 \text{ for all } f \in S\}$. (some books use $Z(S)$). E.g.

- $\Bbbk^n \supset \Bbbk^n = Z(\{0\}) = Z(\emptyset)$;
- $\Bbbk^n \supset \emptyset = Z(\{1\}) = Z(\Bbbk[x_1, \dots, x_n])$.

Remark. System of equations for an algerbaic set X is not unique.

Example 1. $V(z, x^2 + y^2 - 1) = V(z, x^2 + y^2 - 1, 2z, z + (x^2 + y^2 - 1), z(x^3 - y + z))$.

Lemma. *If two functions f_1 and f_2 vanish at some point, then so does $gf_1 + gf_2$ for all $f, g \in \{x_1, \dots, x_n\}$. Put $J =$ the ideal generated by $S \subset \Bbbk[x_1, \dots, x_n]$, then $V(S)$ depends only on J, not on S.*

Corollary. *Hilbert basis theorem guarantees that the polynomial ring is Noetherian, so the ideal J is finitely generated so $V(S) = V(J) = V(\{f_1, \dots, f_m\})$. So every algebraic set is the zero locus of a finite collection*

Example 2. $V(x) = V(x^2)$ both determine the same vanishing set, but $(x) \neq (x^2) \subset \Bbbk[x, y]$.

Theorem (Hilbert's Nullstellensatz)**.** *If a $f|_{V(j)} = 0$ for a (possibly improper) ideal $J \subset \Bbbk[x_1, \dots, x_n b]$, then $f^k \in J$ for some k. (this is strong, requires $\Bbbk$ be algebraically closed)*

Remark. Recall that $\operatorname{rad}(J) = \{f \in R : f^k \in J \text{ for some } k\}$.

(1) $\operatorname{rad} J$ is an ideal;
(2) $\operatorname{rad}\operatorname{rad} J = \operatorname{rad} J$;
(3) $\operatorname{rad} J = \bigcap_{\mathfrak{p} :\, J \subset \mathfrak{p} \subset R} \mathfrak{p}$.

Notation. For a subset $X \subset \Bbbk^n$, set $I(X) \subset \Bbbk[x_1, \dots, x_n]$ the set $\{f : f|_X = 0\}$.

Theorem (Nullstellensatz)**.** $I(V(J)) = \operatorname{rad} J$.

Corollary. *I and V give bijection between algebrais subsets of $\Bbbk^n$ and radical ideals of $\Bbbk[x_1, \dots, x_n]$.*

Example (of Nullstellensatz)**.** $V(J) = \emptyset$ if and only if $1|_{V(J)} = 0$ if and only if $1^k \in J$ if and only if $J = (1)$.
Non-geometric statement: $f_1, \dots, f_k$ have no common zeroes if and only if $\sum g_i f_i = 1$, $g_i \in \Bbbk[x_1, \dots, x_n]$.

Exercise. Properties of V and I.

- $S_1 \subset S_2$ implies $V(S_1) \supset V(S_2)$;
- $X_1 \subset X_2$ implies $I(X_1) \supset I(X_2)$;
- $I(\emptyset) = \Bbbk[x_1, \dots, x_n]$ and $I(\Bbbk^n) = 0$;
- $V(\Bbbk[x_1, \dots, x_n]) = \emptyset$ and $V(0) = \Bbbk^n$;
- $I(V(S)) = \operatorname{rad}\langle S \rangle$;
- $V(I(X))$ is the smallest algebraic set containing X;
- Given any family $S_i \subset \Bbbk[x_1, \dots, x_n]$, we should have $\bigcap V(S_i) = V(\bigcup S_i)$;
- Given S_1 and S_2, $V(S_1) \cup V(S_2) = V(S_1 S_2)$ (finitely many is important).

2. September 6

2.1. **Nullstellensatz.**

Theorem ((Strong) Nullstellensatz)**.** $I(V(J)) = \operatorname{rad} J$

Lemma (Zariski's Lemma (Algebraic Nullstellensatz)**.** *Let $\Bbbk$ be a field (not neceesarily algebraically closed), $K \supset \Bbbk$ a finitely generated algebra that is a field. Then $[K : \Bbbk] < \infty$.*

Example. Take $K = \Bbbk[x]$. This is *not* a finitely generated $\Bbbk$-algebra.

How could it be finitely generated? Take $f_1, \dots, f_m \in K$. If these generate the algebra, then any rational function is of the form $P(f_1, \dots, f_n)$ for some polynomial $P \in \Bbbk[t_1, \dots, t_m]$. But any such polynomial gives a rational function with a particular denominator. This denominator is a product of irreducible factors of denominators of $f_1, \dots, f_m$.

Proof. Suppose K is a transcendental field that is finitely generated $\Bbbk$-algebra. Then $K = \Bbbk(y_1, \dots, y_n)$, and without loss of generality we may assume that K is algebraic over $\Bbbk(y_1)$ (otherwise replace $\Bbbk$ with $\Bbbk(y_1)$).

Then $[K\colon \Bbbk(y_1)] < \infty$, while y_1 is transcendental over $\Bbbk$. Choose a basis $e_1, \dots, e_m$ of K over $\Bbbk(y_1)$. Then $e_i e_j = \sum c_{ij}^k e_k$ for $c_{ij}^k \in \Bbbk(y_1)$.

Take $f_i = \sum g_i^j e_j \in K$ (finite). Then for any polynomial P evaluated like $P(f_1, \dots, f_l) = \sum h_i e_i$, and the only denominators of the finitely many c_{ij}^k and the finitely many coefficients g_i^j contribute to the denominators of h_i. ☐

Corollary. *Suppose $\Bbbk = \bar{\Bbbk}$. Then maximal ideals in $\Bbbk[x_1, \dots, x_n]$ are exactly of the form $(x_1 - a_1, \dots, x_n - a_n) = I($ for some $(a_1, \dots, a_n 0) \in \Bbbk^n$.*

Proof. $J \subset \Bbbk[x_1, \dots, x_n]$ is maximal if and only if $\Bbbk[x_1, \dots, x_n]/J$ is a field, so it is a finitely generated algebra, which implies by Zariski's lemma that the quotient is $\bar{\Bbbk} = \Bbbk$. Hence, $J = \ker(\phi\colon \Bbbk[x_1, \dots, x_n] \to \Bbbk)$ and such a kernel sends $x_i \to a_i$ so kernel is $(x_1 - a_1, \dots, x_n - a_n)$ (which is evidently maximal). ☐

Remark. Same argument shows that if $\Bbbk \neq \bar{\Bbbk}$, then maximal ideals are kernels of evalation homomrphisms onto $\bar{\Bbbk}$. For example, $\mathbb{R}[x, y]/\left\langle x^2 + 1, y^2 + 1\right\rangle = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ which is not a field.

Theorem (Weak Nullstellensatz)**.** *For ideal $J \subset \Bbbk[x_1, \dots, x_n]$ we have $J = \langle 1 \rangle$ if and only if $V(J) = \emptyset$.*

Proof. If this happens, then J is not contained in a maximal ideal of $\Bbbk[x_1, \dots, x_n]$, i.e. not in $(x_1 - a_1, \dots, x_n - a_n)$ for any $a_1, \dots, a_n$, i.e. $(a_1, \dots, a_n) \notin V(J)$ for any $(a_1, \dots, a_n) \in \Bbbk^n$. ☐

Remark. Points of $V(J)$ are in bijection with maximal ideals containing J.

Proof of strong Nullstellensatz (by Rabinowitsch's Trick). Given an ideal $J \subset \Bbbk[x_1, \dots, x_n]$ and $F|_{V(J)} = 0$, we want to show that $F \in \operatorname{rad} J$.

Let $J = (f_1, \dots, f_m)$.

Consider $\Bbbk[x_0, \dots, x_n] \ni f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n), f_0 = x_0 \cdot F(x_1, \dots, x_n) - 1$.

Claim: the system $f_i = 0$ is inconsistent, i.e. there are no solutions.

Why? Because $f_1 = f_2 = \dots = f_m = 0$ we see that $(x_1, \dots, x_n) \in V(J)$, so $F(x_1, \dots, x_n) = 0$, but f_0 is not 0.

By Weak Nullstellensatz, $g_0 f_0 + \sum_{i=1}^m g_i f_i = 1$ where $f_0, g_i \in \Bbbk[x_0, \dots, x_n]$ for all i, but $f_i \in \Bbbk[x_1, \dots, x_n]$ for $i > 0$.

Now, assuming $F \neq 0$, plug in (formally) $x_0 = \dfrac{1}{F[x_1, \dots, x_n]} \in \Bbbk(x_1, \dots, x_n)$.

Then in the $\Bbbk(x_1, \dots, x_n)$, we get $\sum_{i=1}^m g_i(\frac{1}{F}, x_1, \dots, x_n) f_i = 1$. Clearing the denominator, we get some power of F in J. ☐

Remark. Consider $I(V(J))$. We have $(a_1, \dots, a_n) \in V(J)$ if and only if $J \subset (x_1 - a_1, \dots, x_n - a_n)$. Hence, $f \in I(V(J))$ if and only if $f \in \mathfrak{m}$ for all maximal ideals $\mathfrak{m} \subset J$. So $I(V(J)) = \bigcap_{\mathfrak{m} \supset J} \mathfrak{m} = \operatorname{rad} J = \bigcap_{\mathfrak{p} \supset J} \mathfrak{p}$.

Summary: V and I give an inclusion reversing bijectino between radical ideals in $\Bbbk[x_1, \dots, x_n]$ and algebraic subsets $X \subset \Bbbk^n$.

Exercise.

- $V(\sum J_i) = \bigcap V(J_i)$
- $V(J_1 \cdot J_2) = V(J_1) \cup V(J_2)$
- $V(J_1 \cap J_2) = V(J_1) \cup V(J_2)$

Note that even if $\operatorname{rad} J_i = J_i$, then $\sum J_i$ and $J_1 \cdot J_2$ need not be radical, but the intersection is radical. In particular, $\operatorname{rad}(J_1 \cdot J_2) = J_1 \cap J_2$ if J_1 and J_2 are radical.

Example. Let $J_1 = \langle y \rangle, J_2 = \langle y - x^2 \rangle \subset \Bbbk[x, y]$. These are radical because they are principal over square-free (irreducible in fact) polynomials.

$J_1 + J_2 = \langle y, y - x^2 \rangle = \langle y, x^2 \rangle$. Its radical is in fact $\langle x, y \rangle$.

The picture is a parabola and the x-axis intersecting. The sum ideal $\langle y, x^2 \rangle$ remembers that the intersection should be in some sense multiplicty 2 (as it is not transversal).

2.2. Zariski topology.

Definition. The *Zariski topology* on $\Bbbk^n$ is the topology whose closed sets are algebraic sets.

Example. On $\Bbbk^1$ any proper algebraic subset of (algebraically closed) field is finite, so the topology is co-finite.

Remark. Obviously for $\mathbb{C}$ we get fewer open sets than in the usual topology. Also, if $X \subset \Bbbk^n$, then $X \subset V(I(X))$ and the latter is not only the smallest algebraic set containing X, but is the closure of X in the Zariski topology.

Example. So any infinite set is dense in $\Bbbk^1$.

3. September 11

Last time we proved the Nullstellensatz. (Note that in Zariski's lemma, for the choice of basis $\{e_1, \dots, e_m\}$ of $\Bbbk(x_1, \dots, x_n)$ over $\Bbbk(x_1)$, it is important to assume $e_1 = 1$; why?)

We also introduced the Zariski topology on $\Bbbk^n$ in which closed subsets=algebraic subsets. There are very few open/closed sets, so any non-empty open subset is dense.

3.1. Regular function and regular maps. Let $X \subset \Bbbk^n$ be an algebraic set.

Definition. An *algebriac/regular* function $f \colon X \to \Bbbk$ if f is the restriction of a polynomial in $\Bbbk[x_1, \dots, x_n]$.

The *coordinate ring of* X, $\Bbbk[X]$, is the $\Bbbk$-algebra generated by regular functions on X. In particular, $\Bbbk[X] = \Bbbk[x_1, \dots, x_n]/I(X)$.

Example. If $X = \Bbbk^n$, then $\Bbbk[X] = \Bbbk[x_1, \dots, x_n]$. If $X = \emptyset$, $\Bbbk[X] = 0$ (the zero-dimensional $\Bbbk$-algebra)

If $X = V(y) \subset \Bbbk^2$ (the x-axis), then $\Bbbk[X] = \Bbbk[x, y]/(y) = \Bbbk[x]$.

Proposition. $\Bbbk[X]$ is a finitely generated $\Bbbk$-algebra with no nilpotents. Conversely, any finitely generated $\Bbbk$-algebra with no nilpotents is the coordinate ring of some X.

Proof. It follows from the Nullstellensatz. □

Definition. Let X be an algebraic set. For a subset $Y \subset X$, consider $I(Y) = \{f \in \Bbbk[X] \colon f|_Y = 0\}$. This is radical in $\Bbbk[X]$. Conversely, for a subset $S \subset \Bbbk[X]$, consider $V(S) = \{x \in X \colon f(x) = 0 \text{ for all } f \in S\}$. This is an algebraic subset of X.

Theorem. *For any* $J \subset \Bbbk[X]$, $I(V(J)) = \operatorname{rad} J$.

Proof. Exercise. □

Consequently, we have the same bijection between algebraic subests of X and radical ideals of $\Bbbk[X]$. Why? Because $Y \subset X \subset \Bbbk^n$ gives $\Bbbk[x_1, \dots, x_n] \supset I(Y) \supset I(X)$ where $I(Y)$ and $I(X)$ are radical. Again points of X are maximal ideals of $\Bbbk[X]$.

Definition. Let $X \subset \Bbbk^n$ and $Y \subset \Bbbk^m$ be algebraic subsets.

A map $F \colon X \to Y$ is *regular/algebraic* if $F = (f_1, \dots, f_n)$ is such that $f_i \in \Bbbk[X]$ for all i.

In particular, a regular map $X \to \Bbbk^1$ is a regular function.

Example. Let $X = V(y - x^2) \subset \Bbbk^2$ (a parabola). Then the projection (onto the x-axis) $X \to \Bbbk$ defined by sending $(x, y) \to x$ is a regular map. This map is a bijection, with inverse map that happens to also be a regular map as it is given by $t \to (t, t^2)$. We may call such a map biregular.

We actually get a category whose objects are algebraic subsets $X \subset \Bbbk^n$ for any n, and morphisms are regular maps $f \colon X \to Y$.

Corollary. *For any* $F \colon X \to Y$, *we get* $F^* \colon \Bbbk[Y] \to \Bbbk[X]$ *which sends* $f \to f \circ F$ *(a contravariant functor).*

Example.

(1) $f\colon X \to \mathbb{k}^1 = Y$. Then $f \in \mathbb{k}[X]$ so setting $t = f(x_1, \ldots, x_n)$, we get $f^*\colon \mathbb{k}[t] = \mathbb{k}[Y] \to \mathbb{k}[X]$ by $g(t) \to g(f)$.

(2) If $X = \mathbb{k}^0 \to Y$ (a point in Y), then $\mathbb{k}[Y] \to \mathbb{k}$ is evaluation at point of Y.

Theorem. *Any algebra homomoropshim $\mathbb{k}[Y] \to \mathbb{k}[X]$ is of the form F^* for a unique regular map $F\colon X \to Y$.*

Proof. Suppose $Y = V(g_1, \ldots, g_l) \subset \mathbb{k}^m$.

A regular map $F\colon X \to Y$ is given by $(f_1, \ldots, f_m)$ such that $\begin{cases} g_1(f_1, \ldots, f_m) = 0 \\ \vdots \\ g_l(f_1, \ldots, f_m = 0 \end{cases}$.

Given $f_1, \ldots, f_m$ we get $\mathbb{k}[x_1, \ldots, x_n] \to \mathbb{k}[X]$ by $g \to g(f_1, \ldots, f_m)$. Actually, we may as well assume that $I(Y) = \langle g_1, \ldots, g_l \rangle \subset \mathbb{k}[x_1, \ldots, x_m]$. Then the above map has to factor through to $\mathbb{k}[Y] \to \mathbb{k}[X]$. □

Remark. Explicit construction of F given $\phi\colon \mathbb{k}[Y] \to \mathbb{k}[X]$. We have know that maximal ideals of $\mathbb{k}[X]$ are precisely points of X, and maximal ideals of $\mathbb{k}[Y]$ are preciisely points of Y. If $F\colon X \to Y$, then the map from maximal ideals to maximal ideals is $\mathfrak{m} \to \phi^{-1}(\mathfrak{m})$.

Reformulation: we get an anti-equivalence between the category of algebraic sets and the category of finitely generated $\mathbb{k}$-algebras with no nilpotents.

3.2. Affine Varieties.

Remark (Terminology)**.** An affine variety ="abstract algebraic set". An algebraic set=affine variety embedded into $\mathbb{k}^n$.

Definition (Temporary)**.** An *affine variety* is a pair (X, R) where X is a set and R is an algebra of $\mathbb{k}$-valued functions on X such that:

(1) R is finitely generated (no nilpotents because elements are functions to $\mathbb{k}$);

(2) points in X are in one-to-one correspondence with maximal ideals of R.

A *morphism/regular map* of two affine varieties $(X_1, R_1) \to (X_2, R_2)$ is a map $F\colon X_1 \to X_2$ such that $f \circ F \in R_1$ if $f \in R_2$.

Example. If X is an algebraic set, $(X, \mathbb{k}[X])$ is an affine variety.

Remark. Any affine variety is isomorphic to an algebraic set. So we really have an equivalence of categories of algebraic sets, affine varieites, and finitely generated $\mathbb{k}$-algebras with no morphisms (equivalence to $\mathbb{k}$-algebras is an anti-equivalence).

Given R =finitely generated algebra with no nilpotents. Define affine variety as follows. $X = \operatorname{Specm}(R) = \{$maximal ideals of $R\}$. Given a point $\mathfrak{m} \in X$ and an element of R, consider $f + \mathfrak{m} \in R/\mathfrak{m}$. Then by Zariski's lemma, the quotient if actually $\mathbb{k}$, so we can define that to be the value of f at $\mathfrak{m}$.

4. September 13

4.1. Affine varieties revisited.

Last time: A map of algebraic sets $F\colon X \to Y$ is *algebraic/regular* if its components are polynomials. The main result is that we can associate to every algebraic variety X its coordinate ring $\mathbb{k}[X] = \{$regular functions $X \to \mathbb{k}^1\}$ which gives an anti-equvialence between the category of algebraic sets and the category of finitely generated $\mathbb{k}$-algebras without nilpotents (Note that this has a name: it is the functor represented by $\mathbb{k}^1$).

More abstractly, an affine variety is a pair (X, R) where $R \subset \text{Functions}(X \to \mathbb{k})$ with R finitely generated and X in natural bijective correspondence with maximal ideals of R (i.e. such that if $x \leftrightarrow \mathfrak{m}$, then $f(x) = f \bmod \mathfrak{m} \in R/\mathfrak{m} \cong \mathbb{k}$).

To realize (X, R) as an algebraic sset, choose generators $h_1, \ldots, h_m \in R$; relations beteen h_i's form an ideal $J \subset \mathbb{k}[x_1, \ldots, x_n]$. Then we have a surjective $\mathbb{k}$-algebra homomorphism $\mathbb{k}[x_1, \ldots, x_n] \to R$, which factors to give an isomorphism of $\mathbb{k}[x_1, \ldots, x_n]/J \cong R$. Then $X \cong V(J) \subset \mathbb{k}^n$.

Example. $R = \mathbb{k}[t]$. Using the single generator $x = t$, then $X = \mathbb{k}^1$ with coordinaes t.

Take non-efficient generator set $x = t$, $y = t^2$. Then $Y = V(y - x^2) \subset \mathbb{k}^2$ ($\langle y - x^2 \rangle$ is the ideal of relations). This shows that the line is the same as the parabola.

Remark (Notation.)**.** Specm(R) is the set of maximal ideals in R (R finitely generated with no nilpotents). We actually consider Specm(R) as an affine variety in the obvious way.

Traditionally, $\mathbb{A}^n = (\Bbbk^n$ considered as an affine variety$) = \text{Specm}(\Bbbk[x_1, \dots, x_n])$. Philosphically, it is more important that $\mathbb{A}^n$ is a ring of polynomials, it is usually not as important what the underlying field is, which is why often one does not write $\mathbb{A}^n_{\Bbbk}$.

Why is the definition of an affine variety as (X, R) we gave makeshift? It presumes that geometry is *space*=set and a class of functions). But often you do not have global functions on a space, e.g. $\mathbb{P}^1$ (Riemann sphere) has no global holomorphic functions that are not constant.

So a better definition would be *space*=(topological space and locally defined class of functions). The good abstraction of locally defined class of functions is a i*sheaf of functions*. It is worthwhile to look this up as we will eventually go over it and use it, but quickly.

4.2. **Zariski topologoy on algebraic set.** Note: most of what we do will holds for affine varieties (but we need this to properly define the notions of locally defined function).

Definition. The *Zariski topology* on an algebraic set X is the topology whose clased subsets are algebraic subsets $Y \subset X$,i.e. sets cut out by radical ideals of the coordinate ring $\Bbbk[X]$.

Exercise. A regular map between algebraic sets is continuous in the Zariski topology. A biregular map, for example, will be a homeomorphism.

The key property of a coordinate ring is that $\Bbbk[X]$ are Noetherian, so ideals in $\Bbbk[X]$ satisfy the ascending chain condition. Hence, closed subsets in the Zariski topology satisfy the descending chain condition.

Definition. A topological space is *noetherian* if closed subsets satisfy the descending chain condition, i.e. any ($X_0 \supset X_1 \supset \dots$ stabilizers), i.e. if open subsets satisfy the ascending chain condition.

Proposition. Any algebraic set is Noetherian in the Zariski topology.

Proof. Clear. □

Example. On $\mathbb{A}^1$, any descending chain of finite sets stabilizes.
In the classical topology, $\mathbb{R}^n$ is *not* noetherian unless $n = 0$.

Remark (Fact.)**.** A space X is Noetherian and Hausdorff only if X is finite.

Lemma. *Any Noetherian topological space is quasi-compact (compact without Hausdorff), i.e. any open cover has a finite subcover.*

Proof. If $X = \bigcup U_\alpha$, take $U_{\alpha_1} \subset (U_{\alpha_1} \cup U_{\alpha_2}) \subset \dots$. This chain of open sets stabilizes. □

Corollary. *Any algebraic set is quasi-compact in Zariski topology.*

Second proof of Lemma. $X = \bigcup U_\alpha$, then $U_\alpha X - V(J_\alpha)$, and $\emptyset = \bigcap_\alpha V(J_\alpha) = V(\sum J_\alpha)$ if and only if $1 \in \sum_\alpha J_\alpha$, but then 1 belongs to some finite sum of ideals J_α. □

The second proof does not use Hilbert's basis theorem, i.e. does not use the Noetherian condition on the ring. Hence this can be extended to affine schemes whereas the first proof cannot.

Definition. A *principal open set of* X is a set of the form $D(f) = X - V(f) = \{x \in X : f(x) \neq 0\}$ for $f \in \Bbbk[X]$.

Lemma. *The principal open sets form a basis of the Zariski topology.*

Proof. This is a purely formal statement since:

(1) $X - V(J) = \bigcup_{f \in J} D(f)$ (can take finite unions if using Hilbert's Basis theorem and have Noetherian condition).
(2) Intersections are an easy exercise. □

Remark. $\dfrac{f}{g}$ will be defined on $D(g)$, where $f, g \in \Bbbk[x]$ is perhaps one reason to care for open sets.

Definition. A topological space X is said to be *irreducible* if $X \neq \emptyset$ and whenever $X = X_1 \cup X_2$ for closed X_1, X_2, then either X_1 or $X_2 = X$. (if this union is disjoint, then this becomes the definition of a connected space).

Remark. $X \neq \emptyset$ is irreducible if and only if any two non-empty open subsets interect, if and only if every non-empty open set is dense. If X is non-empty and Hausdorff, then X has to be a single point.

Example. $X = \mathbb{A}^1$ is irreducible.

$V(xy) \subset \mathbb{A}^2$ is the union of $V(x)$ and $V(y)$, hence is reducible. The picture is the union of the coordinate axes.

Also $X = \mathbb{A}^n$ is irreducible by the theorem that follows.

Theorem. *An algebraic set X is irreducible if and only if $\Bbbk[X]$ has no zero-divisors, i.e. is a domain.*

Proof. X is irreducible if and only if any two non-empty open subsets intersect. Hence, it is enough to check this property for two principal open sets $D(f)$ and $D(g)$ where $f, g \in \Bbbk[X]$. But $D(f) \cap D(g) = D(fg)$, and $D(f) = \emptyset$ if and only if $f = 0$, so we are done. □

Remark (Reformulation)**.** If $X \subset \mathbb{A}^n$ (or $X \subset Y$), then X is irreducible if and only if $I(X) \subset \Bbbk[x_1, \dots, x_n]$ (or $\subset \Bbbk[Y]$) is prime.

Hence, for algebraic sets Y, by Nullstellensatz we had {closed subsets of Y} corresponding to {radical ideals $J \subset \Bbbk[Y]$}. Points corresponded to maximal ideals, and now irreducible closed subsets correspond to prime ideals.

Theorem. *Any noetherian topological space X can be written as a finite union of closed irreducible sets $X = X_1 \cup \cdots \cup X_m$. Moreover, this is unique up to order if it is irredundant, i.e. $X_i \not\subseteq X_j$ for $i \neq j$.*

Proof of existence. Suppose not. Then X is reducible, so $X - X + 1 \cup X_2 \subsetneq X$ closed. Either X_1 or X_2 cannot be written as a finite union of irreducible sets, say X_1. Then $X \supsetneq X_1$ are two sets for which the statement of the theorem fails, giving us an infinite descending chain of closed subsets, which contradicts the noetherian property. □

Exercise. Show uniqueness.

5. September 18

5.1. **Decompositions into irreducibles.** Last time we have that a topological space is *noetherian* if the closed sets satisfy the descending chain condition, i.e. open sets satisfy the ascending chain condition. We also said that a topological space is *irreducible* if it is non-empty and not a union of proper closed subsets.

Theorem. *Given X =noetherian, there exist finitely many irreducible closed subsets X_i, $i = 1, \dots, m$ such that $X = \bigcup_{i=1}^m x_i$ and $X_i \not\subseteq X_j$ for $i \neq j$. Furthermore, these X_i are unique up to order and they are called the* irreducible components of X.

Remark. The irreducible components of X are precisely the maximal closed irreducible subsets of X.

Corollary. *A noetherian Hausdorff space is finite.*

Proof. The irreducible components are not only irreducible, but also Hausdorff, hence are singletons. □

We have the following:

Proposition. Given X =affine variety with Zariski topology.

Then X is noetheiran (and therefore quasi-compact), and X is irreducible if and only if $\Bbbk[X]$ is integral. Furthermore, irreducible components of X are in bijective correspondence with the minimal prime ideals in $\Bbbk[X]$.

Remark (Algebraic version of the theorem: "weak primary decomposition")**.** In $\Bbbk[X]$, finitely generated $\Bbbk$-algebra, a radical ideal is the intersection of finitely many prime ideals (+uniqueness). This is weaker than Noether-Lasker theorem on primary decomposition.

Example. $\mathbb{A}^n$ is irreducible. Given a hypersurface (*hypersurface*=zero-locus of a single non-zero (non-nilpotent if nilpotents were present?), non-unit function) $V(f) \subseteq \mathbb{A}^n$, we can take f to be square-free, so $\langle f \rangle$ is radical.

Then $V(\langle f \rangle)$ is irreducible if and only if f is irreducible. If $f = f_1 \dots f_k$ is the product of irreducibles, we obtain a decomposition into irreducibles $V(f) = V(f_1) \cup V(f_2) \cup \cdots \cup V(f_k)$. This is true for hypersurfaces of any variety whose coordinate ring is a unique factorization domain.

Example. Closed irreducible subsets of $\mathbb{A}^1$ are $\mathbb{A}^1$ and singletons="irreducible hypersurface" in $\mathbb{A}^1$.

Closed irreducible subsets of $\mathbb{A}^2$ are three types:

(1) $\mathbb{A}^2$ itself;
(2) irreducible plane curves $V(f)$ where $f \in \Bbbk[x, y]$ is irreducible;
(3) singletons.

To explain and do well this classification, we need the notion of dimension, which we will discuss systematically later on. It is not hard to prove by brute force, however (at least the plane case).

Note: we are currently developing the foundational material needed to articulate and prove actual deep theorems, such as the Riemann-Roch theorem for algebraic curves.

5.2. Regular functions on open subsets of (affine variety) X.

Remark (Idea)**.** If we have two reuglar functions, $f, g \in \Bbbk[X]$, we can define $\frac{f}{g} \colon D(g) \to \Bbbk$. We want to say that $\frac{f}{g}$ is regular on $D(g)$.

Example. Consider the circle $X = V(x^2+y^2-1) \subset \mathbb{A}^2$, which we can "parametrize" by $\mathbb{A}^1$ by $(x, y) \to \frac{y}{x+1}$ and $(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}) \leftarrow u$ (note: this fails in characteristic 2, because $X = V(x^2 + y^2 - 1)$ is reducible in characteristic 2).

Now, $\frac{y}{x+1}$ is regular on $X - \{(-1, 0)\}$ while $(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2})$ is regular on $\mathbb{A}^1 - \{\pm i\}$ where $i = \sqrt{-1}$. We want to say that these two open sets are isomorphic, i.e. that the maps restrict to bi-regular maps between the open sets. But what are regular functions on open sets?

(note that we puncture the line, but not at a real point, so real pictures lie!)

Definition. Suppose $f \colon D \to \Bbbk$ is a function on a subset $D \subset X$. We say that f is *regular at* $x \in X$ if there is an open subset $U \subset X$, $U \subset D$, $x \in U$ and $f|_U = \frac{g}{h}|_U$ for $g, h \in \Bbbk[X]$, with h having no zeroes on U.

We further say that f is *regular on an open set* $D \subset X$ if it is regular at all points of D. (Note: it is not obvious that this definition of regular matches the previous definition of regular we had)

Remark.

(1) The condition of f being regular on $D \subset X$ is a local property. There does not necessarily exist $g, h \in \Bbbk[X]$ such that $f|_D = \frac{g}{h}|_D$. For example, if $X = V(y^2 - x^3)$, then $\frac{y+1}{x-1}$ represents a function regular on $X - \{(1, 1), (1, -1)\}$. However, $\frac{y+1}{x-1} = \frac{(y-1)(y+1)}{(y-1)(x-1)} = \frac{y^2-1}{y-1)(x-1)} = \frac{x^3-1}{y-1}x - 1 = \frac{x^2+x+1}{y-1}$. The latter is regular away from $y = -1$, so the two define a regular function on X, which does not have a global representation.

    The problem for the existence of global reprsentation is not singularity of the curve, but lack of unique factorization in $\Bbbk[X]$.

(2) In our definition of *regular at a point*, we may assume that $U = D(\phi)$, where ϕ is a regular function on $\Bbbk[X]$, with $\phi(x) \neq 0$.

    In fact, we can replace $\frac{g}{h}$ by $\frac{g\phi}{h\phi}$, which gives the following definition: f is *regular at* x if there are $g, h \in \Bbbk[X]$ such that $h|_{X-D} = 0$, $h(x) \neq 0$, (so $x \in D(h)$) and $\frac{g}{h} = f$ on $D(h)$.

Theorem. *A function $f\colon X \to \Bbbk$ that is regular at all points of X lies in $\Bbbk[X]$.*

Proof. The proof is easy, but uses a neat trick.

Our assumption is that for every $x \in X$, there is $D(h) \ni x$ such that $f = \dfrac{g}{h}$ on $D(h)$ (where g, h depend on x). We get an open cover of X, and since X is quasi-compact, there exist a finite collection $\{(g_i, h_i)\}$, $i = 1, \ldots, m$ such that $X = \bigcup_{i=1}^m D(h_i)$ and $f = \dfrac{g_i}{h_i}$ on $D(h_i)$. Then $h_i f = g_i$ on $D(h_i)$. We can assume that in fact $h_i = f g_i$ on all of X since the identity fails only on the complement of $D(h_i)$, so we can replace $\dfrac{g_i}{h_i}$ with $\dfrac{g_i h_i}{h_i^2}$.

Hence, since $X = \bigcup_{i=1}^m D(h_i)$ is the same as saying that $\bigcap_{i=1}^m V(h_i) = \emptyset$, so $V(h_1, \ldots, h_m) = \emptyset$ and hence $\langle h_1, \ldots, h_m \rangle = R$, i.e. $1 = \sum_{i=1}^m \phi_i h_i$ for some $\phi_i \in \Bbbk[X]$.

But now $f = f \cdot 1 = f(\sum_{i=1}^m \phi_i h_i = \sum_{i=1}^m \phi_i h_i f_i = \sum_{i=1}^m \phi_i g_i \in \Bbbk[X]$. □

Remark. The key point in the proof is writing 1 as the sum of terms non-trivial on only finitely many open sets in a cover, i.e. we have an "algebraic partition of unity".

Theorem (Algebraic version of theorem)**.** *Let R be a commutative ring (with 1). Given $h_1, \ldots, h_m \in R$ with $\langle h_1, \ldots, h_m \rangle = R = \langle 1 \rangle$ and elements $\psi_i \in R_{h_i}$ (localization of R at $\{h_i^n : n \geq 0\}$), $i = 1, \ldots, m$. We can assume that $\psi_i = \dfrac{g_i}{h_i^{r_i}}$.*

Since we cannot talk about a function f, we can talk about $\dfrac{g_i}{h_i} = \dfrac{g_j}{h_j}$ agreeing on $D(h_i) \cap D(h_j)$. Concretely, we can require that $\psi_i = \psi_j$ as elements of $R_{h_i h_j}$.

Then there exists a unique $\Psi \in R$ such that $\Psi = \psi_i$ in R_{h_i} for all $i = 1, \ldots, m$.

Proof. This is trickier because of possible nilpotents, and the strange notion of equality in localizations.

We can write $\psi_i = \dfrac{g_i}{h_i^{r_i}}$; we can replace h_i with r_i (note: $\langle 1 \rangle = \langle h_1^{r_1}, h_2^{r_2}, \ldots, h_m^{r_m} \rangle$. Then $\psi_i = \dfrac{g_i}{h_i}$.

Then $\psi_i = \psi_j$ means that $\dfrac{g_i}{h_i} = \dfrac{g_j}{h_j}$, which means precisely $(h_i h_j)^{k_{ij}}(g_i h_j - g_j h_i) = 0$. We can assume $k_{ij} = 0$ after $g_i \to g_i h_i^N$ and $h_i = h_i h_i^N$ for $N >> 0$.

Then $g_i h_j - g_j h_i = 0$ and we can write $1 = \sum \phi_i h_i$. Setting $\Psi = \sum \phi_i g_i$, then $\Psi = \dfrac{g_i}{h_i}$ since $\Psi h_i = h_i \sum_j \phi_j g_j = \sum_j \phi_j (g_i h_i) = g_i 1 = g_i$. □

6. September 20

Last time: X =affine variety, and we said that $f\colon U \to \Bbbk$ for a (Zariski) open $U \subset X$ is *regular* if f is locally of the form $\dfrac{g}{h}$, $g, h \in \Bbbk[X]$. Our main fact from last time is that this definition is consistent in the sense that a function is regular on all of X if and only if $f \in \Bbbk[X]$. We also had an algebraic version of this theorem.

6.1. Subvarieties.

Definition. A *subvariety X of $\mathbb{A}^n$* is a subset that is open ins some aglebraic set Y (equivalently: X is open in $\bar{X}$, i.e. is locally closed (the intersection of an open and closed subset).

If $X_1, X_2 \subset \mathbb{A}^n$ are both locally closed and $X_1 \subset X_2$, i.e. X_1 is locally closed in X_2, we say that X_1 is a *subvariety of X_2*.

We have defined what it means for a function $f\colon X \to \Bbbk$ (where $X \subset \mathbb{A}^n$ is a subvariety) to be regular (e.g. X is an open subset of some aglebraic set Y, and this does not actually depend on the choice of Y [I think this is almost obvious: if $f = \dfrac{g}{h}$, for $g, h \in \Bbbk[\bar{X}]$, then it equals $\dfrac{g}{h}$ for $g, h \in \mathbb{A}^n$ and hence in $\Bbbk[Y]$ for any Y containing X).

Definition. A map $X_1 \to X_2$ between subvarieties $X_1 \subset \mathbb{A}^n$, $X_2 \subset \mathbb{A}^m$ is *regular* if its components are regular functions.

This extends our main category {locally closed subsets, regular maps} $\supset$ {closed subsets, regular maps}. The latter is a full subcategory of the former (there are no extra morphisms between closed subsets when we pass to locally closed subsets).

Example. $(x,y) \to \dfrac{y}{x+1}$ is regular on $X - \{1,0\}$ where $X = V(x^2+y^2-1)$, while the inverse $\left(\dfrac{1-u^2}{1+u^2}, \dfrac{2u}{1+u^2}\right) \leftarrow u$ is regular on $\mathbb{A}^1 - \{\pm 1\}$. So the two subvarieties are isomorphic.

Example. Consider the cuspidal cubic $X = V(y^2 - x^3)$. The homework gave a map $t \to (t^2, t^3)$ from $\mathbb{A}^1 \to X$, which is not a regular map. The inverse map is actually $\dfrac{y}{x} \leftarrow (x,y)$, which is not regular. But we can embrase this situation, and see that it is regular on $X - \{0\}$. So now we actually have an isomorphism $\mathbb{A}^1 - \{0\} \cong X - \{(0,0)\}$.

The following example is the simplest, and hence probably the most important.

Example. Consider $Y = V(xy-1)$, the usual hyperbola. We have a map $Y \to \mathbb{A}^1$ sending $(x,y) \to x$ which is obviously regular. Solving for y, we get $(x, \dfrac{1}{x}) \leftarrow x$ which is defined on $\mathbb{A}^1\{0\}$, so that the hyperbola is isomorphic to $\mathbb{A}^1 - \{0\}$.

This is important because $\mathbb{A}^1 - \{0\}$ is now abstractly isomorphic to an affine variety (i.e. in the category of locally closed subsets it is isomorphic to a closed set)!

Proposition. Suppose X is an affine variety, and $f \in \mathbb{k}[X]$. We can define $Y \subset X \times \mathbb{A}^1 = \{(x,y) \colon y \cdot f(x) = 1\}$. This Y is certainly an affine variety (by the homework $X \times \mathbb{A}^1$ is affine, $Y \subset X \times \mathbb{A}^1$ is closd, so affine as well). Then $Y \cong D(f) \subset X$, which in one way is $(x,y) \to x$ and the other way is $(x, \dfrac{1}{f}) \leftarrow x$.

(Rabinowitch's trick is saying that f vanishing on X means $D(f)$ is empty, so Y is empty).

Furthermore, and most impotantly, the principal open set $D(f)$ is an affine variety, because $D(f) \cong Y$ in the category of locally closed sets whose morphisms are regular functions.

So we should have $\mathbb{k}[D(f)] \cong \mathbb{k}[Y]$, but what is $\mathbb{k}[D(f)]$? Well, $\mathbb{k}[X \times \mathbb{A}^1] = \mathbb{k}[X][y]$, so $\mathbb{k}[Y] = \mathbb{k}[X][y]/_{\langle yf-1\rangle}$. This is cheating a bit since we must check that $\langle yf-1\rangle$ is radical, but this construction is just localization! So $\mathbb{k}[X][y]/_{\langle yf-1\rangle} = \mathbb{k}[X]_f$.

Corollary. *Regular functions $D(f) \to \mathbb{k}$ are of the form $\dfrac{g}{f^r}$ where $g \in \mathbb{k}[x]$, $r \geq 0$.*

Exercise. If $D(f) = D(\tilde{f})$, then $\mathbb{k}[X]_f = \mathbb{k}[X]_{\tilde{f}}$.

To summarize: another reason to love principal open sets is that they are secretely affine. But they also form a basis for the Zariski topology! (strange remark: affine sets are analgous to contractible sets that we usually like to use to give bases for topologies). Hence, any open subset $U \subset X$ (X can be affine or locally closed, or anything really) for can be covered by affine open sets (the principal opens!).

6.2. **An aside.** Suppose R is a commutative ring, the $h_1, \dots, h_m \in R$ generate $\langle 1\rangle$, i.e. $1 \in \langle h_1, \dots, h_m\rangle$, and $\psi_i \subset R_{h_i}$ $(i = 1, \dots, m)$ such that $\psi_i = \psi_j$ in $R_{h_i h_j}$, then there exists a unique $\Psi \in R$ such that $\Psi = \psi_i$ in R_{h_i}.

So if $R = \mathbb{k}[X]$, then this is a "gluing statement" for regular functions with respect to the cover $X = \bigcup_{i=1}^m D(h_i)$. This is another point of view: before this showed us regular functions as fractions of global functions were well-defined; now we can think that forcing regular functions to be the localizations on principal opens generates exactly the same thing.

More importantly, the "gluing statement" is the axioms of a sheaf.

Another way of phrasing this is that the sequence $0 \to R \to \prod_i R_{h_i} \to \prod_{i<j} R_{h_i h_j}$ with maps $f \to (f_1, f_2, \dots, f_m)$ and $(f_1, \dots, f_m) \to (f_i - f_j)_{i<j}$ is exact.

There is actually a longer sequence $0 \to R \to \prod_i R_{h_i} \to \prod_{i<j} R_{h_i h_j} \to \prod_{i<j<k} R_{h_i h_j h_k} \to \dots$ (which is a finite sequence; the new maps give alternating signs). Fact: this sequence is exact.

Geometrically: if $R = \mathbb{k}[X]$, this sequence computes the Čech cohomology of X with coefficeints in regular functions with respect to $X = \bigcup_{i=1}^n D(h)i)$. This is why we shouldn't laugh when saying that $D(f)$ are contractible!

Fact: Affine varieties have no higher cohomology with coefficients in regular functions.

6.3. **Varieties.** The idea is the following: abstract algebraic vareites are defined by "gluing" affine charts. Two interpretations: actually glue charts, or just a topological space with a sheaf of ringed functions.

Note that we actually did something today: we made a non-trivial extension by moving to subvarieties.

Remark. Not every localy closed set is affine.

Exercise. For example, $\mathbb{A}^2 - \{(x, y)\}$. One can prove (Homework#2) that any regular function is a restriction from $\mathbb{A}^2$. Then $\mathbb{A}^2 - \{(x, y)\}$ has a maximal ideal that does not correspond to a point in $\mathbb{A}^2 - \{(x, y)\}$.

7. September 25

Last time we talked about the category of locally closed sets, and showed that $D(f) \subset X$ for a variety X is affine, i.e. $\Bbbk[D(f)] = \Bbbk[X]_f$ is an affine variety. Thus every open subset of X can is covered by affine open sets. Our goal is to extend these notions so that we can talk about open subsets as geomeric objects.

7.1. (Abstract) algebraic varieties.

Definition (Classical: with charts)**.** A *vaiety* X is a topological space together with an open cover $X = \bigcup_\alpha U_\alpha$ and homeomorphisms $\theta_\alpha U_\alpha \cong Y_\alpha \subset \mathbb{A}^{n_\alpha}$ where Y_α are algebraic sets (closed in Zariski topology) such that:

(0) The transition functions $\theta_\beta \circ \theta_\alpha^{-1}$ are regular.
(1) X is quasi-compact
(2) X is separated (to be given later, analagous to Hausdorff property)

A *pre-variety* is a topological space satisfying only the first two properties. It is easier to develop the theory for pre-varieties first, and then state separatedness in that language.

Example. Any locally closed subset X of $\mathbb{A}^n$ is a pre-variety (in fact a variety once we know what separtedness is). This is because X is an open subset of an affine (algebraic) $Y \subset \mathbb{A}^n$, hence is covered by principal open sets of Y.

More concrete example: $\mathbb{A}^2 - \{(0, 0)\}$. It is covered by $U_1 = \mathbb{A}^2 - V(x)$ and $U_2 = \mathbb{A}^2 - V(y)$. We did the isomorphisms last time: $U_1 = \mathbb{A}^1 - V(x) \cong V(ux - 1) \subset \mathbb{A}^3$ and $U_2 \cong V(vy - 1) \subset \mathbb{A}^3$. The transition function is $(x, y, u) \to (x, y, \frac{1}{y})$ which is regular on $Y_1 - V(y) \cong Y_2 - V(x)$.

Definition. A *morphism of pre-varieties* $\phi\colon X \to X'$ is a continuous map that is regular on charts: i.e. $\theta'_\beta \circ \phi \circ \theta_\alpha^{-1}$ is regular on its natural domain.

This classical definition is nice for computations, but is not very canonical as it depends on the choice of charts.

7.2. Ringed spaces.

Definition (More abstract)**.** A *pre-variety* is a quasi-compact topological space X together with a sheaf of $\Bbbk$-algebras ("sheaf of regular functions") that is locally isomorphic to an affine variety.

Since we will be working with varieties, rather than schemes, we will use a more concrete version of this definition (i.e. one in which it is obvious that $\Bbbk$-algebras are actually algebras of functions).

Definition. A *sheaf of functions* $\mathcal{O}_X$ on X assigns to any open $U \subset X$ a $\Bbbk$-algebra $\mathcal{O}_X(U) \subset \{$functions $U \to \Bbbk\}$ such that whenever $U = \bigcup_\alpha U_\alpha$, we have $f \in \mathcal{O}_X(U)$ if and only if $f|_{U_\alpha} \in \mathcal{O}_X(U_\alpha)$ for all α.

Remark (Exercise)**.** If $V \subset U$, and $f \in \mathcal{O}_X(U)$, then $F|_V \in \mathcal{O}_X(V)$. (proof: $U = U \cup V$)

Remark (Terminology)**.** $\mathcal{O}_X$ =structure sheaf of X (or of $(X, \mathcal{O}_X)$) and $f \in \mathcal{O}_X(U)$ we call *regular functions*.

Example. When X =topological/differentiable/analytic manifold, then $\mathcal{O}_X(U)$=continuous/differenitable/analytic functions $U \to \Bbbk$ ($\Bbbk = \mathbb{R}$ or $\mathbb{C}$).

If $X \subset \mathbb{A}^n$ is an algebraic set or locally closed subset, then $\mathcal{O}_X(U)$ =regular functions $U \to \Bbbk$

We want to study pairs $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of functions. This unfortunately has no name, as "ringed space" refers to the general pair $(X, \mathcal{O}_X)$ where $\mathcal{O}X$ is a sheaf of rings whose elements (sections) are not necessarily functions.

Definition. A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $\Phi\colon X \to Y$ of topological spaces such that for any $V \subset Y$, and any $f \in \mathcal{O}_Y(V)$, the composition $f \circ \Phi \in \mathcal{O}_X(\Phi^{-1})$ (in varieties, this is precisely requiring that not only components are regular, but that all pullbacks (not just of particular choice of generators) is regular).

Exercise. This gives the usual class of morphisms for topological/differentiable/abalytic manifolds and for locally closed subsets of $\mathbb{A}^n$.

Example. Given $(X, \mathcal{O}_X)$ and open $U \subset X$, define $\mathcal{O}_U$ by $\mathcal{O}_U(V) = \mathcal{O}_X(V)$ since $V \subset U \subset X$ are inclusions of open sets. Then $(U, \mathcal{O}_U)$ is an open subspace of $(X, \mathcal{O}_X)$, and $\phi\colon U \to X$ is a morphism of ringed spaces (it is common to elide the sheaf like that).

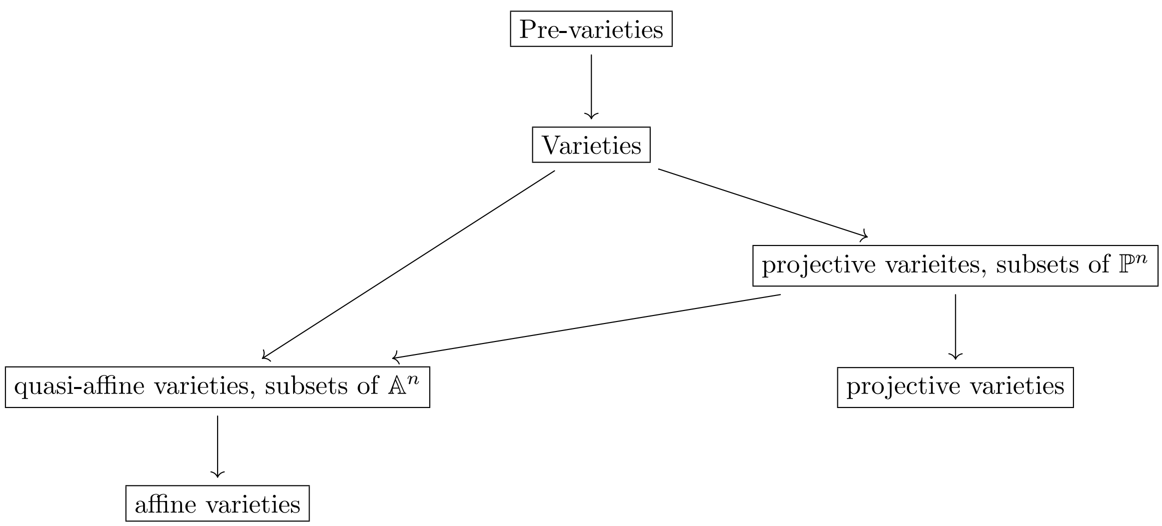
7.3. **Pre-varieties, again.**

Definition. A *pre-variety* is a quasi-compact ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to an algebraic subset in some $\mathbb{A}^n$, i.e. there is a cover $X = \bigcup U_\alpha$ such that $(U_\alpha, \mathcal{O}_{U_\alpha}) \cong$algebraic subset in $\mathbb{A}^{n_\alpha}$ with its natural sheaf of regular functions.

This differs from the previous definition in that the role of the "charts" is not to (locally) equip X with the structure of varieties, but to identify the structure given by the sheaf on X as (locally) the structure of an algebraic variety.

Example. Any locally closed subset of $\mathbb{A}^n$ equipped with its natural sheaf of regular functions becomes a pre-variety.

Remark (Terminology)**.** An *affine variety* is a pre-variety that is isomorphic to a Zariski-closed subset of some $\mathbb{A}^n$. A *quasi-affine variety* is a pre-variety that is isomorphic to a locally closed subset of some $\mathbb{A}^n$.



Definition. A pre-variety X is *separated* (and then a *variety*) if for any pre-variety Y and any regular maps $f, g\colon Y \to X$, the set $\{y\colon f(y) = g(y)\}$ is closed in Y.

It is enough to check just affine Y's.

Example.
(1) If $X = \mathbb{A}^1$, and Y =affine, then $V(f - g)$ is the equalizer set.
(2) If $X = \mathbb{A}^n$ is a variety, then $V(f_1 - g_1, f_2 - g_2, \ldots, f_n - g_n)$ is the equalizer set of $f, g\colon Y \to \mathbb{A}^n$ relying on the fact that they are given as component regular functions.
(3) This works for any locally closed $X \subset \mathbb{A}^n$, so all quasi-affine varieties are indeed varieties.

Example (Non-separated)**.** Consider $\mathbb{A}^1 - \{0\} \subset Y_1 = \mathbb{A}^1$ and $\mathbb{A}^1 - \{0\} \subset Y_2 = \mathbb{A}^1$ glued on $\mathbb{A}^1 - \{0\}$. This is known as $\mathbb{A}^1$ with a doubled point.

Then $\mathbb{A}^1 \cong Y^1 \hookrightarrow X$ and $\mathbb{A}^1 \cong Y_2 \hookrightarrow Y$...

8. September 27

Last time we defined a pre-variety X to be a quasi-compact topological space X with an open cover $X = \bigcup_\alpha U_\alpha$ with $\theta_\alpha \colon Y_\alpha \cong Y_\alpha$ where Y_α is an algebraic set, with a condition that the transition maps are biregular.

Another definition we gave was that X is a quasi-compact topological space X with a sheaf of functions $\mathcal{O}_X$ that is locally isomorphic to algebraic subsets with their sheaf of regular functions.

Exercise. The two definitions are equivalent.

8.1. **Sub(pre-)varieties.** Let $(X, \mathcal{O}_X)$ be a ringed space, $Y \subset X$ a subset with induced topology. Equip Y with the following $\mathcal{O}_Y$: For any open $V \subset Y$, $f \colon V \to \Bbbk$ is regular (so $f \in \mathcal{O}_Y(V)$) if it locally extends to a regular function on X, i.e. if any $y \in V$ has a neighborhood $U \subset X$ containing y and there is $\tilde{f} \in \mathcal{O}_X(U)$ such that $\tilde{f}|_{U\cap V} = f|_{U \cap V}$.

It is clear that so-defined $\mathcal{O}_Y$ is a sheaf of functions on Y. This is the smallest sheaf such that $Y \hookrightarrow X$ is regular (morphism of ringed spaces).

Proposition. If X is a pre-variety and $Y \subset X$ is locally closed, then Y is a pre-variety.

Idea of proof. Reduce to a closed subset $X \subset \mathbb{A}^n$, then check that we get the usual structure sheaf on locally closed $Y \subset X \subset \mathbb{A}^n$. □

Definition. In the above situation (Y is locally closed in X), we say that Y is a *sub-pre-variety* of X. If Y is actually open/closed, we say that Y is an open/closed sub-pre-variety of X.

Example (of using the terminology)**.** We can reword the definitions of affine variety and quasi-affine variety as follows.

An affine variety is a pre-variety isomorphic to a closed subvariety of $\mathbb{A}^n$ (admits a closed embedding into $\mathbb{A}^n$).

Also a quasi-affine variety is a pre-variety isomorphic to a locally closed subvariety of $\mathbb{A}^n$.

8.2. **Product of pre-varieties.**

Remark. If $X \subset \mathbb{A}^n$ and $X' \subset \mathbb{A}^{n'}$ are algebraic, then $X \times X' \subset \mathbb{A}^{n+n'}$ is algebraic.

Definition (Really a construction)**.** Suppose we are given two pre-varieties X and X', consider affine charts $X = \bigcup_\alpha U_\alpha$ and $X' = \bigcup_\beta U'_\beta$ with homeomorphisms $\theta_\alpha \colon U_\alpha \cong Y_\alpha$ and $\theta'_\beta \colon U'_\beta \cong Y'_\beta$.

We construct their *product* $X \times X'$ as $X \times X' = \bigcup_{(\alpha,\beta)} U_\alpha \times U'_\beta$ with $\theta \colon U_\alpha \times U'_\beta \cong Y_\alpha \times Y'_\beta$ given by $\theta = \theta_\alpha \times \theta'_\beta$.

Exercise. $X \times X'$ is a pre-variety.

Remark (Warning)**.** The topology on $X \times X'$ is not the product topology. For example, in $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$, the product topology is the product of co-finite topologies, so the base for the product topology is the union of complemenets of horizontal and vertical lines.

So to get the topology on $X \times X'$, just give the charts the topology for which $U_\alpha \times U'_\beta$ are homeomorphic to $Y_\alpha \times Y'_\beta$, i.e. define $U \subset X \times X'$ to be open if $\theta_\alpha \times \theta'_\beta(U \cap (U_\alpha \times U'_\beta)) \subset Y_\alpha \times Y'_\beta$ is open in the Zariski topology.

Proposition. For pre-varietes Y, X, X' a map $\phi \colon Y \to X \times X'$ is given as (ψ, ψ') is a morphism (regular) if and only if both of the components ψ and ψ' are morphisms (regular).

Proof. Clear. □

Corollary. *$X \times X'$ is the categorical product in the category of pre-varieties, so is unique up to unique isomorphism. In particular, our construction of $X \times X'$ is independent of the choice of charts.*

(This is true because our category of pre-varieties over an algebraically closed field has their regular maps determined set-theoretically on points)

Exercise. If $Y \subset X$ and $Y' \subset X'$ are sub-pre-varieties, then $Y \times Y' \subset X \times X'$ is a sub-pre-variety.

8.3. **Seperatedness revisited.** Recall that given a fixed X, for any pre-variety Y and any morphisms $f, g\colon Y \to X$, we require that $\{y\colon f(y) = g(y)\}$ is closed in Y.

Consider $\phi = (f, g)\colon Y \to X \times X$. Then this set is $\phi^{-1}(\Delta)$ which we require to be closed where $\Delta \subset X \times X$ is the diagonal. This being true for the identity map requires that Δ must be closed, and Δ is closed implies $\phi^{-1}(\Delta)$ is closed since ϕ is continuous. Thus we have proved that:

Proposition (Reformulation)**.** A pre-variety X is separated if and only if $\Delta \subset X \times X$ is diagonal.

Remark. Δ is always locally closed. So we require that $X \hookrightarrow X \times X$ is not just an embedding but a closed embedding.

Recall that a topological space X is Hausdorff if and only if $\Delta \subset X \times X$ is closed with $X \times X$ having the product topology.

So "separated"="Hausdorff" using Zariski topology on $X \times X$.

Exercise.

(1) The product of varieties is a variety.
(2) Sub-pre-variety of a variety is a variety.

(E.g. $\mathbb{A}^n$ is separated implies anything quasi-affine is separated).

8.4. **Rational functions and rational maps.** Suppose X is an irreducible variety. What is good about this is that any two non-empty open sets intersect, so we can compare regular functions on different open sets: if $U \to \Bbbk$ and $U' \to \Bbbk$, we can compare them even if $U \neq U'$!

Example. In $X = \mathbb{A}^1$, we can compare $f_1 = \dfrac{g_1}{h_1}$ and $f_2 = \dfrac{f_2}{g_2}$, but on $X = Z(xy)$ we cannot compare $\dfrac{1}{x}$ and $\dfrac{1}{y}$.

Proposition. Note that if $f\colon U \to \Bbbk$ and $f'\colon U' \to \Bbbk$, for open $U, U' \subset X)$ coincide on some non-empty open set $V \subset U \cap U'$ $f|_{U\cap U'} = f'|_{U\cap U'}$

Proof. Since $U \cap U'$ is irreducible (if $U \cap U' = Y \cup Y'$ where each of Y and Y' is closed, then $X = \bar{Y} \cup Y' \subseteq \bar{(}Y \cup Y'))$ Once we know $f = f'$ on V, it also holds on $\bar{V} = U \cap U'$ (since the variety is separated, the equalizer set is closed). ☐

Same argument as above shows that given f, U, U', there is at most one $f'\colon U' \to \Bbbk$ such that $f|_{U\cap U'} = f'|_{U\cap U'}$ (at most one way to extend f to $U \cup U'$) $(f' - g'|_{U\cap U'} = 0$ implies $f' - g$ is 0 on $U')$.

Definition. Consider the sets $\{(U, f)\colon U \subset X \text{ open}, U \neq \emptyset, f\colon U \to \Bbbk \text{ is regular}\}$.

Then define a relation by (U, f) (U', f') if $f|_{U\cap U'} = f'|_{U\cap U'}$. This is an equivalence relation by the previous proposition. We define a *rational function on* X to be an equivalence class of this relation.

9. October 2

Last time we looked at two important constructions. First: a locally closed subset of a (pre-)variety is naturally a (pre-)variety. Second: the product of (pre-)varieties is naturally a (pre-)variety. Note that the Zariski topology of the product has more open sets than the product topology.

9.1. **Rational functions.**

Definition. A rational function on an irreducible variety X is an equivalence class of pairs (U, f) where $U \subset X$ is a non-empty open subset, and $f\colon U \to \Bbbk$ is a regular function, with (U, f) (U', f') if $f|_{U\cap U'} = f'|_{U\cap U'}$.

It is easy to see that rational functions form a $\Bbbk$-algebra, and in fact a field (inverse of f is defined not necessarily on the domain of f, but on a potentially smaller open subset). We denote it $\Bbbk(X)$.

Proposition.

(1) $\Bbbk(X) = \Bbbk(U)$ for any open non-empty $U \subset X$.
(2) if X is affine, then $\Bbbk(X)$ is the field of fractions of $\Bbbk[X]$ (since X is irreducible, the latter is a domain).

In particular: $\Bbbk(X) \supset \Bbbk(X)$ is always a finitely generated field extension of $\Bbbk$. Conversely, any finitely generated extension of $\Bbbk$ is $\Bbbk(X)$ for some irreducible variety X. Specifically, if $\Bbbk \subset \Bbbk[x_1, \dots, x_n]$, consider $\Bbbk[x_1, \dots, x_n] \subset \Bbbk(x_1, \dots, x_n)$, so $X = \operatorname{Specm} \Bbbk[x_1, \dots, x_n]$.

Remark. For any $f \in \Bbbk(X)$, there is a unique maximal open set U such that f is regular on U, i.e. such that (U, f) is maximal in the equivalence class. We call U the *domain of regularity of f*.

Remark (Homework)**.** $\mathcal{O}_x =$"local ring of x in X" is the stalk of $\mathcal{O}_X$ of $x \in X$. For an irreducible variety the definition can be made simpler: $\Bbbk(X) \supset \mathcal{O}_x = \{f \in \Bbbk(x) \colon f \text{is regular at } x\}$. If X is affine, then $\dfrac{g}{h}$, $g, h \in \Bbbk[X]$, $h(x) \neq$, i.e. $h \notin \mathfrak{m}_x$ where $\mathfrak{m}_x \subset \Bbbk[X]$ is the maximal ideal of the point x. So $\mathcal{O}_x = \Bbbk[X]_{\mathfrak{m}_x}$ (the localization of the ring $\Bbbk[X]$ at the maximal (prime) ideal $\mathfrak{m}_x$).

Thus we can tautologically restate the domain of regularity of $f \in \Bbbk(X)$ as $\{x \colon f \in \mathcal{O}_x \subset \Bbbk(X)\}$ (for all irreducible varieties X).

Example. If X =affine, then the Nullstellensatz says that $\bigcap_{x \in X} \mathcal{O}_x = \Bbbk[X]$.

9.2. Rational maps.

Definition. Suppose that X and Y are irreducible varieties (we really just need X irreducible pre-variety and Y separated).

A *rational map* $f \colon X \to Y$ is an equivalence class of pairs (U, f) where $U \subset X$ is non-empty open subset, where $f \colon U \to Y$ is a regular map, and (U, f) (U', f') if $f|_{U \cap U'} = f'|_{U \cap U'}$ (the equivalence relation uses only that X is irreducible and Y is separated).

If we want to compose, we have a problem: $X \overset{f}{\overset{\text{rational}}{\longrightarrow}} Y \overset{g}{\overset{\text{rational}}{\longrightarrow}} Z$, then $g \circ f$ might not be defined as a rational map because f might avoid the domain of regularity of g.

Definition. A rational map (f, U) $X \overset{\text{rational}}{\longrightarrow} Y$ is *dominant* if $f(U)$ is desnse in Y.

Exercise. The above is independent of U.

If $f \colon X \overset{\text{rational}}{\longrightarrow} Y$ is dominant and $g \colon Y \overset{\text{rational}}{\longrightarrow} Z$ is rational, then $g \circ f \colon X \overset{\text{rational}}{\longrightarrow} Z$ makes sense. In particular, if $g \in \Bbbk(Y)$, then $g \circ f \in \Bbbk(X)$, so we obtain $f^* \colon \Bbbk(Y) \hookrightarrow \Bbbk(X)$.

(of course, the non-dominant rational maps are actually constants)

Remark. Suppose that $f \colon X \overset{\text{rational}}{\longrightarrow} Y$. We can actually assume that we have $f \colon U \to V$ a regular map from an affine open of X to an affine open of V. Then we have $\Bbbk[U] \overset{f^*}{\longleftarrow} \Bbbk[V]$. When is f dominant, i.e. when is $f(\bar{U}) = V$? Well, $U \subset U$ is a closed set given by the (0) ideal of $\Bbbk[U]$. Hence, $f(\bar{U})$ is given by $(f^*)^{-1}(\langle 0 \rangle) \subset \Bbbk[V]$. We require that $(f^*)^{-1}(\langle 0 \rangle) = \langle 0 \rangle$, so f^* has to be injective.

So if this is the case, we certainly have an induced map $f^* \colon \Bbbk(V) \hookrightarrow \Bbbk(U)$, as expected.

Proposition. Conversely, any embedding $\Phi \colon \Bbbk[Y] \to \Bbbk[X]$ equals f^* for unique dominiant rational map $f \colon X \overset{\text{rational}}{\longrightarrow} Y$.

Proof. Without loss of generality, we may assume X and Y are affine: $\Bbbk[Y] = \Bbbk[y_1, \dots, y_n]$ where y_i's are coordinates on $Y \subset \mathbb{A}^n$. Then $y_i \in \Bbbk(Y)$.

Look at $\Phi(y_i) \in \Bbbk(X)$. Shrinking X we may assume that $\Phi(y_i) \in \Bbbk[X]$. Then $\Phi \colon \Bbbk(Y) \to \Bbbk(X)$ extends to $\Bbbk[Y] \to \Bbbk[X]$. Hence $\Phi(\Bbbk[Y]) \subset \Bbbk[X]$. Then Φ comes from a regular $f \colon X \to Y$, necessarily dominant. □

Remark (Summary)**.** The category of irreducible varieties with rational dominant maps is anti-equivalent to the category of finitely generated field extensions of $\Bbbk$.

Remark (Terminology)**.** The isomorphisms in this category are called *birational maps*.

9.3. Examples.

Example. Consider $X = V(x^2 + y^2 - 1) \subset \mathbb{A}^2$ and $Y = \mathbb{A}^1$. Then we have $X \overset{u = \frac{y}{x+1}}{\overset{\text{rational}}{\longrightarrow}} Y$ is a birational map, so $\Bbbk(X) = \Bbbk(x)[y]/_{\langle x^2 + y^2 - 1 \rangle} \cong \Bbbk(u)$.

Hence the unit circle is rational (birational to the line).

Example. Consider $X = V(\langle x^3 - y^2 \rangle)$. Then $(u^2, u^3) \leftarrow u$ that maps $X \leftarrow \mathbb{A}^1$ is regular and birational as $(x, y) \stackrel{\text{rational}}{\longrightarrow} \dfrac{y}{x}$. So the cuspidal cubic is rational.

Example. If $U \subset X$ is non-empty open, then $U \hookrightarrow X$ is bi-rational (that's why $\Bbbk(X) = \Bbbk(U)$) (of course that's circular).

Proposition. *X and Y are birational if and only if there are non-empty open subsets U of X and V of U such that $U \cong V$.*

Proof. Well, for one direction X is birationally isomorphic to U, which is isomorphic to V, which is birationally isomorphic to Y, so X and Y are birationally isomorphic.

Suppose that $f\colon X \stackrel{\text{rational}}{\longrightarrow} Y$ and $g\colon Y \stackrel{\text{rational}}{\longrightarrow} X$ are inverses.

Note that f must be regular on some $U \subset X$ and g on some $V \subset Y$, so $g \circ f$ must be regular on $f^1(V) \cap U$, so $g \circ f = \mathrm{id}$ on that open set, so the image of the composition actually lands in $U \cap f^{-1}(V)$ and f lands in $V \cap g^{-1}(U)$.

Therefore, $f(U \cap f^1(V)) \subset V \cap g^{-1}(U)$. Moreover it is in $V \cap g^{-1}(U \cap f^{-1}(V))$ so we have maps to and from $U \cap f^1(V)$.

□

10. October 4

Last time: we have a rational map $f \stackrel{\text{rational}}{\longrightarrow} X \to Y$ is a regular map from non-empty open $U \to Y$. It is dominant if $f(U) \subset Y$ is dense.

Thus we obtain an anti-equivalence between {irreducible varieties with dominant rational maps}*and*{finitely generated field extensions $K \supset \Bbbk$}.

Example. Start with X an irreducible variety. Suppose its field of ratonal functions is $K \supset \Bbbk$, which is of course a finitely generated field extenision. Choose $x_1, \ldots, x_n \in K$ that are algebraically independent/$\Bbbk$ (over $\Bbbk$) such that $K \supset \Bbbk[x_1, \ldots, x_n]$ is a finite extension (i.e. we chose a transendence basis) so $n = tr\deg(K/\Bbbk)$ (transendence degree of the field extension).

So we have the primitive element theorem when K is separable, so $K = \Bbbk(x_1, \ldots, x_n)[y]$. This is not a problem if $char\Bbbk$ is 0, but there is no reason for the extension to be separable when characteristic is non-zero. However: by choosing carefully the generators, we can ensure the extension is separable.

So now, we can let $f \in \Bbbk(x_1, \ldots, x_n)[t]$ be the minimal polynomial of $y/\Bbbk(x_1, \ldots, x_n)$ (/ means over in this context). But we can actually choose $f \in \Bbbk[x_1, \ldots, x_n, t]$ which will then be irreducible, so $K \cong \Bbbk(X)$ where $X = V(f) \subset \mathbb{A}^{n+1}$.

Corollary. *Any irreducible variety is birationally isomorphic to a hypersurface.*

Proof that we can choose correct transcendental basis above. Let $K = \Bbbk(z_1, \ldots, z_n)$. Assume z_i's are not independent: then there is $f \in \Bbbk[t_1, \ldots, t_n]$ such that $f(z_1, \ldots, z_n) = 0$. We can assume f is irreducible, so $f \notin (\Bbbk[t_0, \ldots, t_n])^p = \Bbbk[t_1^p, \ldots, t_m^p]$ where p is the characteristic $\Bbbk$ (because $\Bbbk$ is perfect). So this means that f has a monomial t_m which appears in f not as t_m^p, so then z_m is separable over $\Bbbk(z_1, \ldots, z_{m-1})$. □

10.1. Dimension.

Definition. If X is irredubcle, we define the *dimension of* X to be $\dim X =$transcendence degree $\Bbbk[X]$.

If $X = \bigcup_i X_i$ with the X_i =irreducible components, then we define the dimension of X to be the largest of the dimensions of the components: $\dim X = \max_i \dim X_i$.

If all components have the same dimension, we say that X has *pure dimension.*

For $x \in X$, we say $\dim_x X = \max_{x \in X_i} \dim X_i$ where X_i are irreducible components.

Proposition.
- If irreducible X and Y are rationally isomorphic, then $\dim X = \dim Y$.
- If $X \stackrel{\text{rational}}{\longrightarrow} Y$ is dominant, then $\dim X \geq \dim Y$.
- $\dim \mathbb{A}^n = n$.
- $\dim X \times Y = \dim X + \dim Y$.
- A hypersurface $V(f) \subset \mathbb{A}^n$ has pure dimension $n - 1$ (i.e. pure codimension 1: the notion of codimension is annoying when ambient space is not pure, can defined as minimum of codimensions at points).

- If Y is irreducible, and $X \subsetneq Y$ is a proper closed subset. Then $\dim X < \dim Y$.

  Proof. Suppose that we have a transcendence basis $\{y_1, \ldots, y_n\} \in \mathbb{k}[Y]$ of $\mathbb{k}(Y)$. Then $\mathbb{k}[X] = \mathbb{k}[Y]/I(X)$. Need to show that $y_i|_X$ are algebraically dependent. Since X is not empty, let $f \in \mathbb{k}(X)$ be a non-zero element of $I(X) \subset \mathbb{k}[Y]$. Then f satisfies a polynomial relation $\sum a_i f^i = 0$ for some $a_i \in \mathbb{k}[y_1, \ldots, y_n]$. We can assume that $a_0 \neq 0$ (just factor out f's), so f divides a_0 and therefore $a_0(y_1, \ldots, y_m)|_X = 0$. □

Corollary. *An algebraic set $X \subset \mathbb{A}^n$ is a hypersurface if and only if X has pure dimension $n-1$.*

Proof. We already have that hypersurfaces have dimension $n-1$. Conversely, if suppose first that X is irreducible. Then there is an $f \in \mathbb{k}[\mathbb{A}^n]$ so that $f|_X = 0$. We can assume f is irreducible because X is. Then $X \subset V(f)$, and then $\dim X = \dim V(f)$, then $X = V(f)$. □

Example. If we look at algebraic subsets of $\mathbb{A}^2$, any such algebraic subset is the union of irreducible componenets. The dimension 2 component can only be $\mathbb{A}^2$, the dimension 1 component has to be a hypersurface, so an algebraic curve, and dimension 0 ones are points.

Theorem. *Let X be an irreducible (affine) variety. Suppose taht $f \in \mathbb{k}[X]$ is a non-zero function. Then $V(f) \subset X$ has pure codimension 1.*

Example. Consider $x^2 + y^2 = z^2$ which gives an irreducible hypersurface in $\mathbb{A}^3$ of dimension 2. Then $V(z = \text{constant}) \cap V(x^2 + y^2 = z^2)$ will have dimension 1. But the real picture is misleading because the vertex of the cone actually has more stuff going through it in $\mathbb{C}$, for example.

Proof of theorem. We want to reduce to the case $\mathbb{A}^n$. We will assume, without loss of generality, that $V(f)$ is irreducible (why? Because if $V(f) = \bigcup Y_i$, we can find a point $y_i \in Y_i$ that is not in any other Y_j, so that $y_i \in U \subset X - \bigcup_{j \neq i} Y_i$. Then U is open, so dense, so has the same dimension as X, and $V(f) \cap U = Y_i \cap U$, so we can replace X with U).

Next, we look at Noether normalization lemma which says that there exist $x_1, \ldots, x_n \in \mathbb{k}[X]$ that are algebraiclyly independent so that $\mathbb{k}[X]$ is a finitely generated $\mathbb{k}[x_1, \ldots, x_n]$-module. Geometrically: $\mathbb{k}[x_1, \ldots, x_n] \hookrightarrow \mathbb{k}[X]$ defines a morphism of varieties $\mathbb{A}^n \leftarrow X \colon \pi$ so that π is dominant.

Remark. If $f \colon X \to Y$ is a map of affine varieties, then f is *finite* if and only if $\mathbb{k}[X]$ is finitely generated $\mathbb{k}[Y]$-module. So Noethern normalization lemma says that $X \to \mathbb{A}^n$ is finite.

Further, the generators $x_1, \ldots, x_n$ can always be chosen to be linear combinations of given generators of $\mathbb{k}[X]$. So if $X \subset \mathbb{A}^m$ is algebraic, in the proof of Noether normalization lemma, $\pi \colon X \to \mathbb{A}^n$ is linear (in fact a projection), i.e. a restriction of a linear map from $\mathbb{A}^m$ to $\mathbb{A}^n$.

So we are going to prove that for $Y = V(f)$ we have the following. Given that $\pi \colon X \to \mathbb{A}^n$ and $Y \subset X$, we will show that $\tilde{Y} = \pi(\bar{Y}) \subset \mathbb{A}^n$ is in fact equal to $V(\tilde{f}) \subset \mathbb{A}^n$ so $\dim \tilde{Y} = n-1$. Then $\dim Y \geq n-1$ as $\pi \colon Y \to \tilde{Y}$ is dominant, and since Y is proper, $\dim Y = n-1$.

Next time, we will use integrality to prove this: that product of integral elements is integral and $\mathbb{k}[\mathbb{A}^n]$ is integrally closed. □

11. October 9

Last time we said that if X =irreducible variety, then $\dim X = tr \deg \mathbb{k}(X)$ (transcendence degree of the field extension).

We have two properties:

(1) If $X \subset Y$ (Y is irreducible) and $\bar{X} \neq Y$, then $\dim X < \dim Y$.
(2) If $X \overset{\text{rational}}{\longrightarrow} Y$ is dominant, then $\dim X \geq \dim Y$.

Theorem (Main Thereom)**.** *Suppose X is affine irreducible, and $f \in \mathbb{k}[X] - \{0\}$. Then $V(f) \subset X$ has pure codimension 1 (so every component has codimension 1).*

For the proof the main idea is to reduce to the case where X is the affine space by using the Noether Normalization Lemma.

Lemma (Noether Normalization Lemma)**.** *If $X \subset \mathbb{A}^n$ is affine, there is a dominant map $\nu \colon X \to \mathbb{A}^m$ which is finite, i.e. $\mathbb{k}[X]$ is a finitely generated $\mathbb{k}[\mathbb{A}^m]$-module (in particular, $[\mathbb{k}(X) \colon \mathbb{k}(\mathbb{A}^m)] < \infty$.*

Remark. This ν can be chosen to be a projection that is the restriction of a linear map $\nu\colon \mathbb{A}^n \to \mathbb{A}^m$, when $\Bbbk$ is infinite.

Also, finite maps are automatically closed, so dominant ν is in fact surjective.

11.1. **Proof of main theorem.**

Proof continued. So far we have two steps:

(1) Without loss of generality we may assume that $Y = V(f)$ is irreducible.
(2) Use Noether Normalization Lemma to choose dominant $\nu\colon X \to \mathbb{A}^m$ where $m = \dim X$.

So we need to show that $\nu(\bar{Y}) = V(\tilde{f}) \subset \mathbb{A}^m$ for some $\tilde{f} \in \Bbbk[\mathbb{A}^m] - \{0\}$. This will imply everything because: $\dim V(\bar{Y}) = \tilde{Y} = m - 1$ and since $\nu\colon Y \to \tilde{Y}$ is dominant, then $\dim Y \geq m - 1$, and since Y is a hypersurface, $\dim Y \leq m - 1$.

Geometrically: we have

$$\begin{array}{ccc} Y = V(f) & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \nu \\ \tilde{Y} = V(\bar{Y}) & \longrightarrow & \mathbb{A}^n \end{array}$$

We want to show $V(\bar{Y}) = V(\tilde{f})$ for some $\tilde{f}$.

Algebraically: $\nu^*\colon \Bbbk[A^m] \hookrightarrow \Bbbk[X]$, the latter a finitely generated odule of $\Bbbk[A^m]$. Then $g \in \Bbbk[A^m]$ satisfies $g|_{\tilde{Y}} = 0$ if and only if $g \in \operatorname{rad}\langle f\rangle \subset \Bbbk[X]$. But $\Bbbk[A^n] \subset \Bbbk[X]$, so $g \in \operatorname{rad}\langle f\rangle \cap \Bbbk[A^m]$.

So we want to construct $\tilde{f} \in \Bbbk[\mathbb{A}^n] - \{0\}$ such that $\operatorname{rad}\left\langle \tilde{f}\right\rangle = \operatorname{rad}\langle f\rangle \cap \Bbbk[A^m]$. This will follow the usual going-up-going-down theorems about how chains of ideals in various rings behave.

So we have $\Bbbk[\mathbb{A}^m] \subset \Bbbk(\mathbb{A}^m) \subset \Bbbk[\bar{\mathbb{A}}^m]$ and $\Bbbk[\mathbb{A}^m] \subset \Bbbk[X] \subset \Bbbk(X) \subset \overline{[\Bbbk(\mathbb{A}^m)]}$ (we choose the last embedding).

Now we can take $\tilde{f}$ to be the $Norm_{\Bbbk(X)/\Bbbk(\mathbb{A}^m)} f$. Two ways: the norm is the free term of a minimal polynomial, or the product of certain conjugates (with certain powers, but since we care about radicals, the powers don't matter).

(0) $\tilde{f} \in \Bbbk[A^m]$. Indeed: f is integral over $\Bbbk[\mathbb{A}^m]$ since $f \in \Bbbk[X]$ which is a finitely generated module means that all of f's conjugates are integral, so certainly the norm $\tilde{f}$ is integral over $\Bbbk[\mathbb{A}^m]$. As the latter is integrally closed, we must have $\tilde{f} \in \Bbbk[A^m]$.
(1) $\operatorname{rad}\left\langle \tilde{f}\right\rangle \supset \operatorname{rad}\langle f\rangle \cap \Bbbk[\mathbb{A}^m]$.

Suppose $g \in \operatorname{rad}\langle f\rangle \cap \Bbbk[\mathbb{A}^m]$, so $g \in \Bbbk[\mathbb{A}^m]$ and $g^k = f \cdot h$ for $h \in \Bbbk[X]$. So $\dfrac{g^k}{f} \in \Bbbk(X)$ is integral over $\Bbbk[\mathbb{A}^m]$. Then $\dfrac{g^k}{f^\sigma}$ is integral where f^σ is a conjugate. Hence, $\dfrac{g^N}{\tilde{f}}$ is integral, so it belongs to $\Bbbk(\mathbb{A}^m)$ and hence to $\Bbbk[\mathbb{A}^m]$. So $\tilde{f}$ divides g^N in $\Bbbk[\mathbb{A}^m]$ (i.e. $g^N \in \Bbbk[\mathbb{A}^m] \cdot \tilde{f}$).

(2) $\operatorname{rad}\left\langle \tilde{f}\right\rangle \subset \operatorname{rad}\langle f\rangle \cap \Bbbk[\mathbb{A}^m]$. It is enough to show that $\tilde{f} \in \operatorname{rad}\langle f\rangle$. This is easy because in fact $\tilde{f} \in \Bbbk[X] \cdot f$ because $f^r + a_{r-1}f^{r-1} + \cdots + a_0 = 0$ for $a_i \in \Bbbk[\mathbb{A}^m]$ for a minimal polynomial and $a_0 = \tilde{f}$, so $\tilde{f} \in \Bbbk[\mathbb{A}^m] \cdot f \subset \Bbbk[X] \cdot f$. □

Corollary. *Given Y =irreducible pre-variety, $X \subset Y$ a closed irreducible sub(pre-)variety. For any integer d between* $\dim X$ *and* $\dim Y$*, there is an irreducible closed sub(pre-)-variety $X \subset Z \subset Y$ such that* $\dim Z = d$.

Proof. Replace Y by an affine neighborhood of $x \in X$ (after Z is constructied in this neighborhood, take its closure in Y).

If $X = Y$ there is nothing to prove, otherwise: $X \subset V(f)$ with $f \in \Bbbk[Y] - \{0\}$, and we can replace Y by one of the irreducible components of $V(f)$. □

Definition (Topological definition of dimension in a "noetherian topology")**.** For a pre-variety X, $\dim X = \max\{n\colon X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \subset X$ with all X_i are closed and irreducible$\}$.

Remark. With this definition, the corollary says that given $Y \subset X$ for an irreducible X, we can always refine to a chain that gives the dimension of X.

If X is affine, then $\operatorname{topdim} X$ =Krull dimension of $\Bbbk[X]$.

If R is a finitely genearated algebra with no zero divisors, then we have proven that the Krull dimsnion of R is the transcendence degree of its field of fractions.

Lemma. *Let Y be an affine irreducible (or just affine of pure dimension) with dimension n. Suppose $X \subset Y$ is closed, and also of pure dimension $n-m$ (so pure codimension m).*

Then there exist $f_1, \dots, f_m \in \Bbbk[Y]$ such that X is contained in $V(f_1, \dots, f_m)$ and $V(f_1, \dots, f_m)$ is of pure dimension $n-m$.

Remark (Krull's Height Theorem)**.** All components of $V(f_1, \dots, f_m) \subset Y$ for $f_i \in \Bbbk[Y]$ have dimension $n-m$ or more.

Proof of lemma. We proceed by induction.

For first step, we need to choose f_1 such that $f_1|_X = 0$ and $V(f_1) \subset Y$ has pure codimension $n-1$. It is enough to ensure that f does not vanish on any componenet of Y. Choose a point on each component of Y that is outside of X. Then there is a function $f \in I(X)$, but does not vanish on those points. This is so because $I(X)$ is a vector space, and for every point y_i we have $I(X) \cup \{y_i\}) \subset I(X)$ is a proper vector subspace. Hence, $I(X)$ is not the union of finitely many of those if $\Bbbk$ is infinite, which it is as $\Bbbk = \bar{\Bbbk}$, thus we can find $f \in I(X0 - \bigcup I(X \cup \{y_i\}) \neq \emptyset$. ∎

12. October 11

Homework is due Thursday October 18, NOT Tuesday October 16.

12.1. **Complete intersections.** Last time we proved the key theorem: that if X =irreducible and $f\colon X \to \mathbb{A}^1$ is not zero, then $\{f=0\} \subset X$ has pure codimension 1.

We also proved the lemma that if Y =affine of pure dimension n, and $X \subset Y$ is closed of pure dimension $n-m$, then there exist $f_1, \dots, f_m \in \Bbbk[X]$ such that $X \subset V(f_1, \dots, f_m)$ and such that $V(f_1, \dots, f_m)$ has pure codimension $n-m$. What the lemma says is that X has a bunch of components, some of which are of pure codimension m.

Remark (Terminology)**.**

(1) $X \subset Y$ is a *set-theoretic complete* intersection if there exist $f_1, \dots, f_m$ such that $X = V(f_1, \dots, f_m)$.
(2) $X \subset Y$ is a *(ideal-theoretic or scheme-theoretic) complete intersection* if f_i's can be chosen so that $\langle f_1, \dots, f_m \rangle = I(X) \subset \Bbbk[Y]$.

Example. Let $Y = V(y^2 - x^3)$. Then $X = \{(a^3, a^2)\} \subset Y$ for $a \neq 0$ is not a complete intersection, even set-theoretically.

Easiest way to see this is to choose a parametrization. We have a bjiective regular map $t \to (t^3, t^2)$, which is birational/dominant, thus gives an embedding $\Bbbk[Y] \hookrightarrow \Bbbk[t]$, so the ring is $\Bbbk[Y] = \Bbbk[t^2, t^3]$.

If $f \in \Bbbk[Y]$ vansihes only on X, then $f(t^3, t^2)$ vanishes only at a, so $f(t^3, t^2) = c \cdot (t-a)^n$, but the right-hand side does lie in this ring unless $n = 0$.

Note that single points are contained in sets of finitely many points that are zero-loci, just not exactly in a zero-locus with a single point unless $a = 0$.

Remark. $\Bbbk[Y]$ is a UFD if and only if every hypersurface is a complete intersection.

Theorem (Generalized Key Theorem)**.** *Let $f\colon X \to Y$ be a map between (pre-)varieties of pure dimension. Then for any $y \in f(X)$, $\dim f^{-1}(Y) \geq \dim X - \dim Y$.*

Remark. The estimate is better if f is assumed dominant (replace Y with the closure of $f(X)$).

Proof of theorem. We can assume without loss of generality that both X and Y are affine. For any $y \in Y$, there exist $g_1, \dots, g_n \in \Bbbk[Y]$ where $n = \dim Y$ such that $S = V(g_1, \dots, g)n) \ni y$ is finite.

Then $V(g_1 \circ f, g_2 \circ f, \dots, g_n \circ f)$ =finite union of fibers, including the fiber $f^{-1}(y)$. So every component of $f^{-1}(y)$ has dimension at least $\dim X - n = \dim X - \dim Y$. ∎

Remark (To be proved later)**.** If $f\colon X \to Y$ is dominant, and X, Y are irreducible, then there is a dense open $U \subset Y$ such that for any $y \in Y$, $f^{-1}(y)$ is non-empty and has dimension $\dim X - \dim Y$.

In particular, $f(X)$ has a big locally closed subset.

Example. Consider $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \to (x, yx)$. The image if $\{(x, y)\colon x \neq 0\} \cup \{(0, 0\}$, which is not locally closed.

This is bad in the way that the fiber at the origin is 1-dimension, on the y-axis away from the origin, the fibers are 0-dimensional, but away from the y-axis everything is fine (fibers are 2-dimensional as they should be)???

Main reason not to prove this now, is that Noether Normalization Lemma is essentially about projections. These are best handled in projective spaces.

12.2. **Projective variety.** We let $\mathbb{P}^n$ to be the set of lines through 0 in $\Bbbk^{n+1}$. Homogeneous coordinates $(x_0\colon \ldots : x_n)$ correspond to lines, have equivalence $\Bbbk \cdot (x_0, \ldots, x_n) \neq 0$. This gives $\mathbb{P}^n$ as a set.

Now, $U_i = \{(x_0\colon \ldots : x_n)\colon x_i \neq 0\} \subset \mathbb{P}^n$ is isomorphic to $\mathbb{A}^n = \left\{(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i})\right\}$. Clearly, $\mathbb{P}^n = \bigcup_{i=0}^n U_i$. The transition functions are obviously regular. Hence, $\mathbb{P}^n$ is an algebraic (pre-)variety (will prove is separated next time).

Remark. A *projective variety* is a closed subvariety of $\mathbb{P}^n$, and a *quasi-projective* variety=subvariety of $\mathbb{P}^n$ (this includes all quasi-affine varieties).

Example. Consider $\mathbb{P}^1$. We will think of it as $\mathbb{A}^1 \cup \{\infty\} = U_0 \cup U_\infty$, where $\Bbbk[U_0] = \Bbbk[t]$, then $\Bbbk[U_\infty] = \Bbbk[\frac{1}{t}]$.

Since U_0 and U_1 are both dense in $\mathbb{P}^1$, we get $\Bbbk(\mathbb{P}^1) = \Bbbk(t)$, but $\Bbbk[U_0] \cap \Bbbk[U_\infty] = \Bbbk$. So only regular functions on $\mathbb{P}^1 \to \mathbb{A}^1$ are constants. Hence, $\mathbb{P}^1 \not\leftrightarrow \mathbb{A}^n$ for any n, so $\mathbb{P}^1$ is not affine.

Similarly, any $\mathbb{P}^n \to \mathbb{A}^1$ is constant.

Example. Let $f \in \Bbbk[x_0, \ldots, x_n]$ be a homogeneous poylnomial. Then set $V(f) = \{(x_0\colon \ldots : x_n)\colon f(x_0, \ldots, x_n) = 0\}$. Clearly $V(f)$ is closed in projective space. $V(f) \cap U_i = V(f(a_0, \ldots, a_{i-1}, 1, \ldots, a_{i+1}, a_n))$.

Also clearly, for any set $S \subset \Bbbk[x_0, \ldots, x_n]$ of homogeneous polynomials, $V(S) = \bigcap_{f \in S} v(f) \subset \mathbb{P}^n$ is closed. What is not immediately clear is that the converse is also true: any closed subset $X \subset \mathbb{P}^n$ is of this form.

Proof. Suppose $X \subset \mathbb{P}^n$ is closed. Take $y \in \mathbb{P}^n - X$. There exists a homogeneous f such that $f|_X = 0$ but $f(y) \neq 0$. Indeed, suppose $y \in U_0 \subset \mathbb{P}^n$ with $U_0 = \mathbb{A}^n \ni (a_1, \ldots, a_n)$. Then there exists $g(a_1, \ldots, a_n)$ such that $g|_{X \cap U_0} = 0$.

We homogenzie this polynoimal to get $f(x_0, x_1, \ldots, x_n) = g(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \cdot x_0 \deg(g)$.

The only bad thing that can happen is that at the hyperplane at infininty $x_0 = 0$, f may not behave well, but we addresses that by considering $x_0 \cdot f$.

Now start writing $V(f_1) \supset V(f_1, f_2) \supset V(f_1, f_2, f_3) \supset \cdots$ and us eht noetherian property. ☐

Remark. Another way of stating this claim is that $\mathbb{P}^n - V(f)$ for homogeneous f form a basis for the Zariski topology of $\mathbb{P}^n$. In fact, $U_i = \mathbb{P}^n - V(x_i)$, and even better (but not immediately clear): if $\deg f > 0$, $\mathbb{P} - V(f)$ are affine.

13. October 16

Last time: we discussed the dimension of fibers, and definition of $\mathbb{P}^n \ni (x_0\colon \ldots : x_n)$. We proved that any closed subvariety of $\mathbb{P}^n$ is $V(S)$ for a set $S \subset \Bbbk[x_0, \ldots, x_n]$ of homogeneous polynomials.

13.1. **Seperability of $\mathbb{P}^n$.** The following variations of the previous results are useful:

- A closed subvariety of $\mathbb{A}^n \times \mathbb{P}^m \ni ((x_1, \ldots, x_n), (y_0\colon \ldots : y_n))$ is a zero-locu of a set of polynomials in $\Bbbk[x_1, \ldots, x_n, y_0, \ldots, y_n]$ which are homogeneous in the y's.
- A closed subvariety of $\mathbb{P}^n \times \mathbb{P}^m \ni ((x_0\colon \ldots : x_n), (y_0\colon \ldots : y_m))$ is given by a polynmoials in $\Bbbk[x_0, \ldots, x_n, y_0, \ldots, y_m]$ for homogeneous x' and $y's$.

Example. We will show that $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ is closed. We want to write a condition that $(x_0\colon \ldots x_n) = (y_0\colon y_n)$. This is equivalent to requiring that $\begin{pmatrix} x_0 & \cdots & x_n \\ y_0 & \cdots & y_n \end{pmatrix}$ has rank 1, i.e. that $x_i y_j - y_i x_j = 0$ for any $0 \leq i, j \leq n$.

Corollary. *$\mathbb{P}^n$ is separated, so any quasi-projective pre-variety is separated, so a variety.*

13.2. Projective Closures.

Example. Consider $X = V(x^2 + y^2 - 1) \subset \mathbb{A}^2 \ni (x, y)$. We can consider this as sitting inside $\mathbb{P}^2 \ni (x_0 : x_1 : x_2)$ with $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$.

Thus we get $V(x_1^2 + x_2^2 - x_0^2) \subset \mathbb{P}^2$, determining a projective curve $\bar{X}$.

(We say X is an *affine* quadric, while $\bar{X}$ is a *projective* quadric).

In fact, $\bar{X}$ =irredcuible projective curve, so it is the Zariski closure of X.

Exercise. A closed subvariety of $\mathbb{P}^n$ has pure codimension 1 if and only if $X = V(f)$ for homogeneous f. Furthermore, X =irreducible if and only if f can be taken irreducible.

Example. Take $X : x^2 + y^2 - 1 = 0 \subset \mathbb{A}^2 \ni (x, y)$. We have maps $X - \{(-1, -)\} \ni (x, y) \to \frac{y}{x+1}$ and $(\frac{1-u^2}{u^2+1}, \frac{2u}{u^2+1}) \leftarrow u \in \mathbb{A}^1 - \{\pm i\}$.

If we pass to the projective closures, the following happens.

First, set $x = \frac{x_1}{x_0}$ and $y = \frac{x_2}{x_0}$. This gives $X \subset \bar{X}$. Also let $u = \frac{u_1}{u_0}$, which gives $\mathbb{A}^1 \subset \mathbb{P}^1$.

Thus the above maps become rational maps $\bar{X} \leftrightarrow \mathbb{P}^1$. In homogeneous coordinates, we get $(x_0 : x_1 : x_2) \to \frac{y}{x+1} = \frac{x_2/x_1}{x_1/x_0+1} = \frac{x_2}{x_1+x_0}$. This is all if we just consider as a rational map from $\mathbb{P}^2 \supset \bar{X}$ to $\mathbb{A}^1 \subset \mathbb{P}^1$, but otherwise $\frac{x_2}{x_1+x_0} = (x_1 + x_0 : x_2)$.

For the reverse map we get $(u^1 + 1 : 1 - u^2 : 2u) \leftarrow (u_0 : u_1)$ where $u = \frac{u_1}{u_0}$, so in fact we have $(u_1^2 + u_0^2 : u_0^2 - u_1^2 : 2u_1u_0) \leftarrow (u_0 : u_1)$.

It turns out that the map $\mathbb{P}^1 \to \bar{X}$ is regular because $(u_1^2 + u_0^2 : u_0^2 - u_1^2 : 2u_1u_2) = (0 : 0 : 0)$ only if $(u_1 : u_0) = (0 : 0) \notin \mathbb{P}^1$.

The map $\bar{X} \to \mathbb{P}^1$ is not regular because $(x_1 + x_0 : x_2) = (0 : 0 : 0)$ if $(x_0 : x_1 : x_1) = (1 : -1 : 0)$. It is however rational everywhere because this was only one representation! Another one is $(x_2 : x_0 - x_1)$ since $\bar{X} = V(x_1^2 + x_2^2 - x_0^2)$. Hence we actually have a ratoinal isomorphism $\bar{X} \cong \mathbb{P}^1$.

Exercise. Every irreducible quadric in $\mathbb{P}^2$ is isomorphc to $\bar{X} \cong \mathbb{P}^1$.

Exercise (Homework)**.** Every rational map $\mathbb{P}^1 \to \mathbb{P}^n$ is regular. So any rational map $\mathbb{A}^1 \to \mathbb{P}^n$ is also regular.

13.3. Another point of view.

We have a map $\mathbb{A}^{n+1} - \{0\} \xrightarrow{p} \mathbb{P}^n$.

For any $X \subset \mathbb{P}^n$, $p^{-1}(X)$ is "strech"-invariant ($\mathbb{P}^n = \mathbb{A}^{n+1} - \{0\}/\mathbb{k}^\times$).

Proposition. X is closed if and only if $p^{-1}(X)$ is closed. In particular, the Zariski topology on $\mathbb{P}^n$ is the quotient topology of $\mathbb{A}^{n+1}$.

Proof. One direction is clear by construction of p. For the other, we see that p locally admits a regular section. What does this mean? It means that on each $U_i \subset \mathbb{P}^n$ given by $U_i : x_i \neq 0$ we have an inverse to p sending $(x_0 : \ldots : x_n) \to (\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i})$. This is an isomorphism $U_i \to U_i \times (\mathbb{A}^1 - \{0\})$ with $\mathbb{k}^\times$ acting on $\mathbb{A}^1 - \{0\}$. □

This is an example of a principal $\mathbb{k}^\times$-bundle (aka $\mathbb{k}^\times$-torsor).

So we now have the following correspondence: "closed subsets of $\mathbb{P}^n$"$\leftrightarrow$"$\mathbb{k}^\times$-invariant closed subsets of $\mathbb{A}^{n+1} - \{0\}$"$\to$"$\mathbb{k}^\times$-invaraint closed subsets of $\mathbb{A}^{n+1}$ containing 0".

Ultimately, we get $X\top^{-1}(X) \cup \{0\} = CX$ is the *affine cone* of X. What can we say about $I(CX) \subset \mathbb{k}[x_0, \ldots, x_n]$?

Proposition. Closed $Y \subset \mathbb{A}^{n+1}$ is $\mathbb{k}^\times$-invaraint if and only if $I(Y) \subset \mathbb{k}[x_0, \ldots, x_n]$ is homogeneous.

Proof. If the ideal is homogeneous, it is generated by homogeneous equations, so one direction is clear. For the converse, if $f = \sum_{i=1}^d f_i \in I(Y)$ where f_i are homogeneous of degree i. Since Y is $\mathbb{k}^\times$-invariant, we know that $f^{(\lambda)}(x_0, \ldots, x_n) = f(\lambda x_0, \ldots, \lambda x_n) \in I(X)$.

But span $\langle f^{(\lambda)} \rangle$ = span $\langle f_i \rangle$ by Vandermonde determinant (which requires infinite field! or polynomial in λ at any particular point has to be 0, so its components should be). □

Theorem (Projective Nullstellensatz)**.** *We have a correspondence $X \to I(V)$ and $V(J) \leftarrow J$ where $I \subset \mathbb{k}[x_0, \ldots, x_n]$ and $X \subset \mathbb{P}^n$, which is a bijection between closed subvarieties of $\mathbb{P}^n$ and radical homogeneous proper ideals $J \subsetneq \mathbb{k}[x_0, \ldots, x_n]$, i.e. $J \subset \langle x_0, \ldots, x_n \rangle$, the unique maximal homogeneous ideal.*

Theorem (Projective Weak Nullstellensatz)**.** *For homogeneous ideal $J \subset \Bbbk[x_0, \ldots, x_n]$, $\emptyset = V(J) \subset \mathbb{P}^n$ if and only if $J \supset (x_0, \ldots, x_n)^N$ for some $N >> 0$ (i.e. $(J)_{(N)}$, the degree N homogeneous part of J, contains, i.e. is, $\Bbbk[x_0, \ldots, x_n]_{(N)}$).*

This is much more manageable than the affine case because the two sides of the equality $(J)_{(N)} = \Bbbk[x_0, \ldots, x_n]_{(N)}$are both finite-dimensional vector-spaces.

14. October 18

Last time: we had the projective Nullstellensatz which gave a bijection between closed $X \subset \mathbb{P}^n$ and radical homogenenous ideals $I \subset \langle x_0, \ldots, x_n\rangle \subset \Bbbk[x_0, \ldots, x_n]$.

The weak Nullstellensatz said that given homogeneous ideal $J \subset \Bbbk[x_0, \ldots, x_n]$, then $V(J) = \emptyset$ if and only if $J^{(N)} = \Bbbk[x_0 \ldots, x_n]^{(N)}$ (graded pieces are equal) for $N >> 0$.

14.1. Theorem of the day.

Theorem. *For any $f\colon X \to Y$, with X projective and Y separated, $f(X)$ is closed.*

Corollary.

1. *Such f is closed.*
2. *Suppose $f\colon X \to \mathbb{A}^1$, X is projective. Then $f(X)$ is either finite or $\mathbb{A}^1$. But we can also consder $f\colon X \to \mathbb{A}^1 \subset \mathbb{P}^1$, so then $f(X)$ must be closed in $\mathbb{P}^1$, so cannot be $\mathbb{A}^1$. Hence, $\#f(X) < \infty$.*
   If X is connected and projective, then any regular function on X is constant.
3. *If X both projective and affine, then $\#X < \infty$.*
4. *If locally closed $X \subset \mathbb{P}^n$ is actually projective, then X is closed.*

We will prove the theorem by proving the following proposition:

Proposition. For any pre-variety Y, the projection $\pi_2\colon \mathbb{P}^n \times Y \to Y$ is closed.

The Proposition implies the Theorem. Given $X \subset \mathbb{P}^n$ closed and $f\colon X \to Y$, then $f(X) = \pi_2(\Gamma_f)$ where $\Gamma_f = \{(x, y) \in \mathbb{P}^n \times Y : x \in X, y = f(x)\}$. If Y is separated, then Γ_f is closed.

Hence, $\pi_2(\Gamma_f)$ is closed since the proposition tells us that π_2 is a closed map. □

Proof of Proposition. The key to the proof will be the weak Nullstellensatz.

The statement is local on Y, so we may assume $Y \subset \mathbb{A}^m$. But in fact, we may assume $Y = \mathbb{A}^m$ since $\mathbb{P}^n \times Y \subset \mathbb{P}^n \times \mathbb{A}^m$ gets sent to $Y \subset \mathbb{A}^m$ by the projection map π_2.

Given closed $X \subset \mathbb{P}^n \times \mathbb{A}^m$ and $y \in \mathbb{A}^m - \pi_2(X)$, we need to construct $U \subset \mathbb{A}^m$ containing y such that $U \subset \mathbb{A}^m - \pi_2(X)$.

But we know that $X = V(S)$ where $S \subset \{f \in \Bbbk[x_0, \ldots, x_n, y_1, \ldots, y_m] \colon f$ is homogeneous in x_i's$\}$.

But what is $X \cap \pi_2^{-1}(y)$? We can think of it as $X_y = X \cap (\mathbb{P}^n \times \{y\}) = V(S_y)$ where $S_y = \{f(-, y)\colon f \in S\} \subset \Bbbk[x_1, \ldots, x_n]$.

Therefore, $y \notin \pi_2(X)$ if and only if $X_y = \emptyset$, if and only if for some N, the ideal spanned by S_y contains $\Bbbk[x_0, \ldots, x_n]^{(N)}$.

This is equivalent to requiring that there exists $f_1, \ldots, f_k \in S$ (so $f_1(-, y), \ldots, f_k(-, y) \in S_y$) and homogenenous polynomials $g_1, \ldots, g_k \in \Bbbk[x_0, \ldots, x_n]$ such that $\{f_i(-, y) \cdot g_i\}_{i=1}^k$ is a basis in $\Bbbk[x_0, \ldots, x_n]^{(N)}$.

But the condition that this is a basis gives an open subset of $\mathbb{A}^m$ (some determinant is not equal to zero). □

Remark (Terminology)**.** A variety X such that for any pre-variety Y, the projection $\pi_2\colon X \times Y \to Y$ is closed is called *complete* (some call it *proper*). So we showed that $\mathbb{P}^n$ is complete, and hence any projective variety (closed subset of $\mathbb{P}^n$) is complete.

14.2. Examples of projective varieties.

Example (Segre variety)**.** The Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m}$ is given by $((x_i), (y_i)) \mapsto (x_i y_j)_{\substack{i=0,\ldots,n \\ j=0,\ldots,n}}$ (x_i and y_j are homogeneous coordinates).

It is convenient to index coordinates of $\mathbb{P}^{nm+n+m}$ by two indices, so $\mathbb{P}^{nm+n+m} \ni z_{ij}$. Then the embedding is $z_{ij} = x_i y_j$.

This actually gives us $(l \subset \Bbbk^{n+1}, l' \subset \Bbbk^{m+1}) \mapsto (l \otimes l' \subset \Bbbk^{n+1} \otimes \Bbbk^{m+1})$ where l, l' are lines.

We claim that S is a closed embedding.

Proof. $(z_{ij}) \in S(\mathbb{P}^n \times \mathbb{P}^m)$ if and only if $\mathrm{rk}(z_{ij}) = 1$ which is the closed condition that all 2×2 minors vanished.

Remark. We can prove that the image is closed by remarking that the product of complete varieties is complete, so their image is closed.

To construct the inverse map $s^{-1}\colon s(\mathbb{P}^n \times \mathbb{P}^m) \to \mathbb{P}^n \times \mathbb{P}^m$, note that it is given by $(z_{ij}) \mapsto ((z_{i0}), (z_{0j}))$. This is problematic if one of the columns is identically zero, but we have many representations: $(z_{ij}) \to ((z_{ik}, (z_{lj})$ for any fixed k and l. For every point, at least one of these representations works. □

Example. When $n = m = 1$, $s(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^3$ is given by $V(z_{00}z_{11} - z_{01}z_{10})$ which is a quadric surface in 3-dimensinoal space. So the two families of lines on this quadric are images of the $\mathbb{P}^1 \times \{x\}$ and $\{x\} \times \mathbb{P}^1$.

Corollary. *$\mathbb{P}^n \times \mathbb{P}^m$ is projective. Hence product of projective varieties is projective, and product of quasi-projective varieties is quasi-projective.*

Now some of the problems hard enough to be homework become easy.

Example (Homework)**.** Let V_d =vector space of polynomials of degree $\leq d$. Let $\tilde{V}_d$ be the corresponding projective space (i.e. $V_d - \{0\}/\mathbb{k}^\times$, alternative notation is $\mathbb{P}(V_d)$ or $\mathbb{P}(V_d^*)$).

For $1 \leq d' < d$, consider product map $\tilde{V}_{d'} \times \tilde{V}_{d-d'} \to \tilde{V}_d$. This is a map from a projective variety to a separated variety, so it's closed: hence the condition that polynomials split into a product of two polynomials of degree d' and $d - d'$ is Zariski closed.

Example (Veronese Embedding)**.** The Veronese embedding is $v\colon \mathbb{P}^n \to \mathbb{P}^N$ sending $(x_i) \to$ (all monomials of degree d in the x_i).

Fact: this is also a closed embedding. Linear algbera says that it sends $(l \subset \mathbb{k}^{n+1}) \mapsto (l^d \subset \mathrm{Sym}^d\mathbb{k}^{n+1})$.

Example. If $n = 1$, $d = 2$, then $(x_0\colon x_1) \to (x_0^2\colon x_0x_1\colon x_1^2) = (y_0\colon y_1\colon y_2)$. The only condition is $y_0y_2 - y_1^2 = 0$.

So $\mathbb{P}^1$ is isomorphic to quadric in $\mathbb{P}^2$.

Example (The Grassmanian)**.** $\mathrm{Gr}(k, n)$ =linear spaces $W \subset \mathbb{k}^n$ of dimension k, which is the smae as projective subspaces $\tilde{W} \subset \tilde{V} = \mathbb{P}^{n-1}$ of dimension $k - 1$. We will continue this next time.

15. October 23

Last time there were two important results. The most important result was the completeness property of projective varieties: for projective X and any Y, $\pi_2\colon X \times Y \to Y$ is closed (if Y is separated).

The second important result was the Segre embedding: $s\colon \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m}$.

15.1. Grammansian.

We denote by $\mathrm{Gr}(k, n)$ the k-dimensional subspaces W in n-dimensinoal vector space $\mathbb{k}^n = V$. This is the same as the projective $k - 1$-dimensional subspaces $\tilde{W} \subset \mathbb{P}^{n-1} = \tilde{V}$.

Choosing basis B =$n \times k$ matrices of rank B up to right multiplication by $\mathrm{GL}_{\mathbb{k}}(k)$.

We define $\mathrm{Gr}(k, n)$ as a variety by using charts. Fix a subspace $U \subset V$ with $\dim U = n - k$. Then $\{W \in \mathrm{Gr}(k, n)\colon W \cap U = 0\} \cong \mathbb{A}^{k(n-k)}$.

Choose one W'. Then any such W is a graph of a linear map $W' \to U$. E.g. if $W' = \langle e_1, \ldots, e_k\rangle$, $U = \langle e_{k+1}, \ldots, e_n\rangle$. Then we have a set of matrices B such that $\det(b_{ij})_{i,j=1}^k \neq 0$. Acting by $\mathrm{GL}(\mathbb{k})$, we can make $B = \begin{pmatrix} I \end{pmatrix}$ where $*$ =coordinates on this char are $\mathbb{A}^{k(n-k)}$.

It is not clear from this description exactly what the varietiy is.

Definition (Plücker coordinates)**.** We have a $p\colon \mathrm{Gr}(k, n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$.

This is $W \hookrightarrow \Lambda^k W$ since if $W \subset V$,then $\Lambda^k W \subset \Lambda^k V = \mathbb{k}^{\binom{n}{k}}$.

Explicitly: $B \hookrightarrow (\det(b_{\alpha_i j})_{i,j=1}^k)$ indexed by $1 \leq \alpha_1 < \cdots < \alpha_k \leq n$.

Proposition. p is a closed embedding.

Proof. Consider an affine chart $\mathbb{A}^{\binom{n}{k}-1} \subset \mathbb{P}^{\binom{n}{k}-1}$. Its preimage is one of the carts on $\mathrm{Gr}(k, n)$.

For instance, consider the chart where $\det(b_{ij})_{i,j=1}^k \neq 0$. IN these coordinates, p looks like this:

$B = \big(I\big) \mapsto (\det(b_{\alpha_i j})) = (1\colon \underbrace{*\colon \ldots : *}_{\substack{\text{non-homogeneous} \\ \text{coordinates on } \mathbb{A}^{\binom{n}{k}-1}}})$. They include all the original coordinates.

Up to reindexing, $\bar{x} \mapsto (\bar{x}, F(\bar{x}))$. □

Remark. Using more linear algebra, we can write explicit equations, called Plücker relations, for $p(\mathrm{Gr}(k,n)) \subset \mathbb{P}^{\binom{n}{k}-1}$. They are quadratic.

This is an example of one of my favorite varieties: its points classify something, so the Plücker variety actually answers a classification problem.

15.2. **Incidence Variety.**

Definition. $\mathrm{Inc} \subset \mathbb{A}^n \times \mathrm{Gr}(k,n)$ is defined by $\{(x, W)\colon x \in W\}$. This it the "tautolical (or universal) family of subspaces $W \subset V$".

The above is indeed closed, which is easy to see if we consider the affine charts.

Remark. What is a family? If we have map $X \to Y$, then the fibers form a family parametrized by Y. E.g. $\mathbb{A}^n \times Y \to Y$ is a constant family of vector spaces over Y.

So, family of subvarieties if $X \subset \mathbb{A}^n \times Y$ such that X_y =fiber of X over $y \in Y$ is a subspace of $\mathbb{A}^n$. (The really correct thing to say is "subbundle in a trivial vector bundel").

The universality of Inc means the following: a family X_y like the above defines a map $Y \to \mathrm{Gr}(k,n)$ sending $y \to X_y \subset \mathbb{A}^n$. This is a regular map, so this gives a universal property of the Grassmanian.

Definition. We also have $\tilde{\mathrm{Inc}} \subset \mathbb{P}^{n-1} \times \mathrm{Gr}(k,n)$, where $\tilde{\mathrm{Inc}} = \{(x, \tilde{W})\colon x \in \tilde{W}\}$. Again, this is a closed subvariety, hence a projective variety as both $\mathbb{P}^{n-1}$ and $\mathrm{Gr}(k,n)$ are projective.

Example. If $X \subset \mathrm{Gr}(n,k)$ is closed, then $\bigcup_{W \in X} W \subset \mathbb{A}^n$ is closed. It is $\pi_1(\pi_2^{-1}(X) \cap \mathrm{Inc})$, therefore closed, where:

$$\begin{array}{ccc} & \mathrm{Inc} & \\ & \downarrow & \\ & \mathbb{A}^n \times \mathrm{Gr}(k,n) & \\ {\scriptstyle \pi_2} \swarrow & & \searrow {\scriptstyle \pi_1} \\ \mathbb{A}^n & & \mathrm{Gr}(k,n) \end{array}$$

Example. $\bigcup_{\tilde{W} \in X} \tilde{W} \subset \mathbb{P}^{n-1}$ is closed.

Example. Given closed $Y \subset \mathbb{P}^n$, the set of projective subspaces meeting it will be closed, specifically the set is $\{\tilde{W} \in \mathrm{Gr}(k,n)\colon \tilde{W} \cap Y \neq \emptyset\}$.

Showing its closerd is similar: $\pi_2(\pi_1^{-1}(Y) \cap \tilde{\mathrm{Inc}})$. Note that since π_2 is the projection along the space, we need to take the space to be complete (e.g projective), otherwise this is not closed.

Example. Consider $\{x\} = \{(1\colon 0\colon \ldots : 0)\} \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^{n-1} = \{(0\colon *\colon \ldots : *)\} \subset \mathbb{P}^n$.

The union of all lines joining $\{x\}$ with points of Y is closed in $\mathbb{P}^3$ by the previous examples. This is called the "projective cone of Y".

15.3. **Projections in projective space.** Geometric appraoch: Take $\mathbb{P}^n$, fix a point $x \in \mathbb{P}^n$. Choose a hyperplane H disjoint from $\{x\}$. This hyperplane is isomorphic to $\mathbb{P}^{n-1}$, and for any other point y we consider the line joining x and y and where it intersects H. So a projection is a regular map $\mathbb{P}^n - \{x\} \to H \cong \mathbb{P}^{n-1}$.

Algebraic approach: by a linear change of coordinates, we can assume $\{x\} = \{(1\colon 0\colon \ldots : 0)\}$, and $H = \{(0\colon *\colon \ldots : *)\}$, then $(x_0\colon \ldots : x_n) \mapsto (x_1\colon \ldots : x_n)$. More generally, this is a linear map $\mathbb{k}^{n+1} \to \mathbb{k}^n$.

Third approach: coordinate-free version: $x \in \mathbb{P}^n$ is a line $l \subset V \cong \mathbb{k}^{n+1}$. Then we have a map $\tilde{V} \to (\widetilde{V/l})$.

Fourth approach: Consider in $\mathrm{Gr}(2, n+1)$ (lines in in $\mathbb{P}^n$) the subvariety Y of lines through x. Then the map $\mathbb{P}^n - \{x\} \to Y$ is given by $x \neq y \to$line through x and y. Then a choice of H gives $Y \cong H \cong \mathbb{P}^{n-1}$.

Remark. These projections allow us to study dimensions of projective varieties.

Iterated projection: $(x_0\colon \ldots : x_n) \to (x_m\colon \ldots : x_n)$. This looks like projection from a subspace.

16. October 25

Missing.

17. October 30

Missing.

18. November 1

18.1. **Tangent spaces.** Last time: tangent spaces. We have several approaches (X =affine) to defining it:

(1) $X \subset \mathbb{A}^n$, $T_xX = \{a \in \Bbbk^n \colon \langle df(x), a\rangle = 0 \text{ for all } f \in I(X)\}$.
(2) Derivations $\partial \colon \Bbbk[X] \to \Bbbk$ such that $\partial(fg) = f\partial g + (\partial f)g$.
(3) $T_xX = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$.

Functoraility: X, Y are affine, then $\Phi \colon X \to Y$ induces $d\Phi \colon T_x \to T_yY$ where $y = \Phi(x)$. We can show this in three ways:

(1) If $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^2$, then $d\Phi(x) = (\dfrac{\partial \Phi_i}{\partial x_i}(x))$. This is not uniquely-defined, but its values are uniquely defined.
(2) A derivation $\partial \colon \Bbbk[x] \to \Bbbk$ induces a derivation $d\Phi(x)(\partial) \colon \Bbbk[Y] \to \Bbbk$, which will be $\partial \circ \Phi^*$
(3) $\Phi^* \colon \Bbbk[Y] \to \Bbbk[X]$ satisfies $\Phi^*(\mathfrak{m}_y) \subset \mathfrak{m}_x$ and $\Phi^*(\mathfrak{m}_y^2) \subset \Phi^*(\mathfrak{m}_x^2)$. Then we have an induces map $\Phi^* \colon (\mathfrak{m}_y/\mathfrak{m}_y^2) \to (m_x/m_x^2)$. Then $d\Phi(x)$ is the adjoint (dual).

Remark. $d\Phi(x)$ is defined for any Φ regular on a neighborhood of x. This is clear for defintion 1, but for definitions 2 and 3 they become non-trivial algebraic facts.

Suppose Φ is regular on $U = D(f) = X - V(f)$, and $\Bbbk[U] = \Bbbk[X]_f$. Then $d\Phi(x) \colon T_xU \to T_xY$. We need to show that $T_xU = T_xX$.

For the second definition, this says that derivation $\Bbbk[X] \to \Bbbk$ uniquely extends to $\Bbbk[U] \to \Bbbk$. But elements of $\Bbbk[U]$ are $\frac{g}{f^n}$, so the unique extension will be given by the quotient rule.

For the third defintion, we are looking at $\Bbbk[X]/\mathfrak{m}_x^k \cong \Bbbk[x]_f/\mathfrak{m}_x^k \cdot \Bbbk[x]_f$ for $k \geq 0$ because $f \notin \mathfrak{m}_x$. So f is invertible in $\Bbbk[X]/\mathfrak{m}_x^k$.

Definition. This allows us to define the tangent space at $x \in X$ for any pre-variety X as T_xU for open affine $U \subset X$. We can use the differentials of transition functions to show independence of U.

Remark (Recall)**.** We have $\mathcal{O}_{x,X}$ =local ring of a point. If X is affine, $\mathcal{O}_{x,X} = \Bbbk[X]_{\mathfrak{m}_x}$. We can actually modify definitions 2 and 3 as follows:

(2) $T_xX \ni \delta \colon \mathcal{O}_{x,X} \to \Bbbk$ where $\partial(fg) = (\partial f)g(x) + f(x)\partial g$
(3) $T_xX = (\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2)^*$ where $\tilde{\mathfrak{m}}_x \subset \mathcal{O}_{x,X}$ with $\tilde{\mathfrak{m}}_x = \{f \colon f(x) = 0\}$.

The equivalence of these definitions and the original ones are precisely the argument from the previous remark.

18.2. **Properties of T_xX.**

Proposition.

(1) For any x, $\{x \in X \colon \dim T_xX \geq s\}$ is closed.

Proof. In definition 1, $X \subset \mathbb{A}^n$, T_xX is the zero-set of of $df_i(x)$ where $\langle f_1, \ldots, f_m\rangle = I(X)$. So look at the rank of the matrix $(df_1, df_2, \ldots, df_n)$. Alternatively, $X \to \mathbb{A}^m$ given by $(f_1, \ldots, f_m)$ gives rank of Jacobian... □

(2) If X is irreducible, then $\dim T_xX \geq \dim X$ with equality on a dense open set. We checked this for hypersurfaces, and any X is birational to a hypersurface.

Note: If $X = \bigcup X_i$ where X_i are irreducible components of X. If $x \in X_i$, then $T_xX_i \subset T_xX$ (this holds for any subvariety)

Corollary. $\dim T_xX \geq \dim_x X$. *Furthermore, we have equality on a dense open set (openness is not clear at this point). Fact: equality is only possible if X is irreducible at that point.*

Definition. X is smooth if $\dim T_xX = \dim_x X$ and singular otherwise.

Remark. X is smooth if and only if $\mathcal{O}_{x,X}$ is regular.

18.3. **Geometry of smooth points.** Fix $x \in X$. Consider local parameters at x, i.e. $(t_1, \ldots, t_m)$ regular near x such that they vanish at the point, and form a basis in $\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$, where $\tilde{\mathfrak{m}} \subset \mathcal{O}_{x,X}$. (If $X \subset \mathbb{A}^n$, then we can always choose such local parameters to be linear).

Remark. The image of f in $\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$ is the differential $df(x) \in T_x^*X$.

The local parameters are particularly useful at a smooth point (because then the local parameters will be algebraically independent).

Lemma. $(t_1, \ldots, t_m) = \tilde{\mathfrak{m}}$

Proof. This is Nakayama's Lemma. □

The geometric meaning: If X is smooth at x, then the equations say that $t_1 = t_2 = \cdots = t_m = 0$ has $\{x\}$ as an irreducible component. This is not new. What is new, is that for a small neough open affine $U \ni x$, the maximal ideal of x is generated by exactly m elements, i.e. it is a complete intersection.

So $x \in X$ is a (scheme-theoretic) local complete intersection (l.c.i.) if X is smooth. (Before: we proved that $x \in X$ is a set-theoretic l.c.i., no matter what the variety X is). In fact, $x \in X$ is a an l.c.i. if and only if $x \in X$ is smooth.

Example. Suppose X =curve. Then $x \in X$ is smooth if and only if $\tilde{\mathfrak{m}} \subset \mathcal{O}_{x,X}$ is principal, if and only if $\mathcal{O}_{x,X}$ is a discrete valuation ring.

We say that $0 \neq f \in \mathcal{O}_{x,X}$ is of the form $t^d g$ where g is a unit, i.e. $g(x) \neq 0$ and t is a local parameter, i.e. $\langle t \rangle = \tilde{\mathfrak{m}}$. Then d does not depend on the choice of t, and $d = \mathrm{ord}_x f$.

Example (Application)**.** Any rational map $f \colon X \overset{\text{rational}}{\longrightarrow} \mathbb{P}^n$ is regular at smooth point x.

Proof. $f = (f_0 \colon \ldots \colon f_n)$ near x (if you want, $f_i \in \mathcal{O}_{x,X}$). Then we can always cancel out a large enough power of t to make this regular. In particular, divide all the f_i's by $t^{\min(\mathrm{ord}_{f_i})}$. □

19. November 6

Last time $x \in X$ is smooth if $\dim T_x X = \dim_x X$. When $x \in X$ is smooth, local parameters behave nicely. Concretely, $t_1, \ldots, t_k$ form a basis for $\tilde{m}/\tilde{m}^2$ ($\tilde{m} \subset \mathcal{O}_{x,X}$). This says that the $dt_i(x)$ form a basis in T_x^*X (the cotangent bundle), and $t_i(x) = 0$.

Claim: if $x \in X$ is smooth, then $(t_1, \ldots, t_k)$ for $k = \dim_x X$, then t_i are also local parameters (except they don't vanish at y) for all $y \in U$ for U an open neighborhood of x. This will be easy later on.

Remark. Suppose $x \in X$ is smooth. Then $(t_1, \ldots, t_k)$ is not a chart. Consider the case of a curve: then the local parameter is a map to $\mathbb{A}^1$, but there is no way for there to be an inverse map on some open subset of $\mathbb{A}^1$. Consider a parabola $x = y^2$ mapped to the line by $t = x$. Generically this is a two-to-one map, so there is no way to write an inverse function, and in fact the fields of rational functions are $\Bbbk(\sqrt{t})$ and $\Bbbk(t)$.

The fact that the t is a local parameter, means that $dt(x)$ is bijective, but there is no algebraic inverse function theorem. Algebraically, $\mathcal{O}_{x,X}$ is not necessarily isomorphic to $\mathcal{O}_{0,\mathbb{A}^k}$: smoothness only assures that the local ring is regular. In dimension 1 we have discrete valuation rings, but there's many of them.

Remark (Terminology)**.** Suppose $f \colon X \to Y$ is a morphism with $x \in X$, $y = f(x) \in Y$ are smooth. Then f is *étale* at x if $df(x) \colon T_x X \to T_y Y$ is bijective.

(Over $\mathbb{C}$, where we do have inverse function theorem, étale maps are isomorphisms locally in the analytic sense)

Remark. We say that f is *unramified* if *df* is injective, and that f is *smooth* if *df* is surjective. ("smooth" does not mean differentiable because everything is differentiable,but rather "submersive").

Note that if we have a morphism $f \colon X \to Y$ we think of it as a family of fibers $\{f^{-1}(y)\}_{y \in Y}$. An easy exercise: if f is smooth at x, then $Z = f^{-1}(y)$ is also smooth at x.

19.1. **Taylor series.** Throughout we suppose $x \in X$ is a smooth point with local parameters $(t_1, \dots, t_k)$. Recall that by Nakayama's lemma, $\langle t_1, \dots, t_k \rangle = \tilde{\mathfrak{m}}$. Then degree d monomials in $(t_1, \dots, t_k)$ generate $\tilde{\mathfrak{m}}^d = \langle t_1^d, t_1^{d-1} t_2, \dots, \rangle$.

Proposition. Degree d monomials in t_i's form a basis $\tilde{\mathfrak{m}}^d / \tilde{\mathfrak{m}}^{d+1}$

Proof. Let $p(z_1, \dots, z_k)$ be a degree d homogeneous polynomial which is not identically zero. We need to show that $p(t_1, \dots, t_k)$ is not contained in $\tilde{\mathfrak{m}}^{d+1}$.

We may without loss of generaliy assume that $p(1, 0, 0, \dots, 0) = 1$ (after a linear change of variables: if A is an invertaible matrix, then $A \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} \bar{t}_1 \\ \bar{t}_2 \\ \vdots \\ \bar{t}_n \end{pmatrix}$ give new local parameters $(\bar{t}_1, \dots, \bar{t}_k)$)).

Then $p(t_1, \dots, t_k) = t_1^d$+other terms.

Now consider $\mathcal{O}_{x,X} / _{\langle t_2, \dots, t_k \rangle} = \mathcal{O}_{x,Y}$ where Y =zero locus of $(t_2 = \cdots = t_k = 0)$ on X. We know that $\dim_x Y \geq 1$ (by dimension of fibers). But maximal ideal in $\mathcal{O}_{x,Y}$ is generated by a single element t_1, so $\dim T_x Y \leq 1$. So $\dim_x Y = 1 \dim T_x Y = 1$ and Y is smooth at x. Furthermore, $\mathcal{O}_{x,Y}$ is a discrete valuation ring.

We have thus reduced the problem to the case where the local ring is a DVR. In $\mathcal{O}_{x,Y}$, $t_1^d \notin \langle t_1^{d+1} \rangle$. Hence, $t_1^d \notin \langle t_1^{d+1}, t_2, \dots, t_k \rangle \subset \mathcal{O}_{x,X}$, so $p(t_1, \dots, t_k) \notin \langle t_1^{d+1}, \dots, t_2, \dots, t_k \rangle \supset \tilde{\mathfrak{m}}^{d+1}$. □

Corollary. *$\mathcal{O}_{x,X} / \tilde{\mathfrak{m}}^{d+1}$ ahs a basis formed by monimla sin t_i's of degree $\leq d$, i.e. $\mathcal{O}_{x,X} / \tilde{\mathfrak{m}}^{d+1} \cong \mathbb{k}[t_1, \dots, t_k] / \mathfrak{m}^{d+1}$ where $m = \langle t_1, \dots, t_k \rangle \subset \mathbb{k}[t_1, \dots, t_k]$.*

This corresponds to computing the taylor polynomial of $f \in \mathcal{O}_{x,X}$ using local parameters.

Another way to see this is that the local parameters give a map t from an open set around $x \in X$ to $\mathbb{A}^k$ sending $x \to 0$. This is NOT an isomorphism $\mathcal{O}_{0,\mathbb{A}^k} \to \mathcal{O}_{x,X}$ but it is an isomorphism modulo $\tilde{\mathfrak{m}}^d$, i.e. $\mathbb{k}[\mathbb{A}^k] / \tilde{\mathfrak{m}}^{d+1} = \mathcal{O}_{0,\mathbb{A}^k} / \tilde{\mathfrak{m}}^{d+1} \cong \mathcal{O}_{x,X} / \tilde{\mathfrak{m}}^{d+1}$.

Exercise: this holds for any étale map.

19.2. **Taylor series.** Consider $\hat{\mathcal{O}}_{x,X} = \varprojlim \mathcal{O}_{x,X} / \tilde{\mathfrak{m}}^d = \{$sequences of $a_i \in \mathcal{O}_{x,X} / \tilde{\mathfrak{m}}^i$ such that $a_i = a_{i+1}$ mod $\tilde{\mathfrak{m}}^i\}$. This is also the completion of $\mathcal{O}_{x,X}$ inthe $\tilde{\mathfrak{m}}$-adic topology (basis of opens is $g + \tilde{\mathfrak{m}}^d$).

Theorem. *If $x \in X$ is smooth, then $\hat{\mathcal{O}}_{x,X} \cong \mathbb{k}[[t_1, \dots, t_k]]$.*

Remark (Fact)**.** For tautological reasons, there is a natural map $\mathcal{O}_{x,X} \to \hat{\mathcal{O}}_{x,X}$ which is the Taylor series map. This map is injective (algebraically $\bigcap_k \tilde{\mathfrak{m}}^k = 0$; this should follow from Nakayama's lemma, but not obviously).

Corollary. *(1) $\mathcal{O}_{x,X}$ is a domain.*
(2) t_i's are algebraically independent.

The first means that a smooth point lies on a single irreducible compoenent (relies on injectivity of the map). The second relies on the existence of the map.

Remark (Conrete approach to the Theorem)**.** Assume without loss of generality that $X \subset \mathbb{A}^n$ (statement of theorem is local). In fact, near $x \in X$, X can be given by $n - k$ equations $f_1, \dots, f_{n-k}$ such that the Jacobian has maximal rank (this is smoothness). So choose k variables complementary to a maximal non-vanishing minor. Then these k variables will be local parameters on X, and the remaining variables will be formal power series in terms of them.

This is a formal implicit function theorem.

20. November 8

20.1. **Singularities.** Last time $x \in X$, smooth implies that $\hat{\mathcal{O}}_{x,X} \cong \mathbb{k}[[t_1, \dots, t_k]]$ where the t_i's are local parameters at x. Generally, for singular points, $\hat{\mathcal{O}}_{x,X}$ determines the "formal type" of the singularity.

Example. Consider $y^2 = x^3 + x^2$ and $y^2 = x^2$ where characteristic is not 2.

First, the local rings at the singularities are not isomorphic (because one singularity is reducible, the other one is not). The formal completions of the local rings, however, are isomorphic. This formal completition is $\Bbbk[[x, y]]/\left\langle y^2 - (x^3 + x^2(\right\rangle$ and $\Bbbk[[u, v]]/(u^2 - v^2)$. The isomorphism is $y = v$ and $u = \sqrt{x^3 + x^2}$.

Completing a ring can introduce zero-divisors (forgets about irreducible components)

Remark. Note that $\hat{\mathcal{O}}_{x,X} = \varprojlim \mathcal{O}_{x<X}/\mathfrak{m}^k = \varprojlim \Bbbk[X]/\mathfrak{m}^k = \varprojlim \Bbbk[x, y]/(y^2-(x^3+x^2))/\mathfrak{m}^k = (\varprojlim \Bbbk[x, y]/\mathfrak{m}^k)/(y^2-(x^3 + x^2))$.

Remark. $x \in X$ is smooth if and only if $\hat{\mathcal{O}}_{x,X} \cong \Bbbk[[t_1, \dots, t_k]]$.

Geometrically: $\hat{\mathcal{O}}_{x,X}$ are "functions on the formal neighborhood of x". If x is smooth we call it the formal (poly)disk.

Example. Solving linera algebraic ODE's like $f' = f$ has no non-trivial algebraic solutions (in $\mathcal{O}_{x,X}$, but we have solutions in $\hat{\mathcal{O}}_{x,X}$).

Theorem. *Suppose $x \in X$ is smooth. Then $\mathcal{O}_{x,X}$ is a UFD.*

Purely algebraic proof. Eisenbud-Harris. □

Proof from Shafarevich. Step 1: Prove that $\Bbbk[[t_1, \dots, t_k]]$ is a UFD. How? Well, for polynomials: Gauss's lemma allows for an induction to work. For formal power series, there is Weierstrass Preparation Theorem.

If $f \in \Bbbk[[t_1, \dots, t_k]]$ is such that $0 \neq f(0, 0, \dots, 0, t)) \in \Bbbk[[t]]$. We can assume $f(0, 0, \dots, 0, t_k) = at_k^n$+higher order terms. We can write this as t_k^n(unit).

Weierrstass Prepartion THeorem says that there exists a uniue $g \in \Bbbk[[t_1, \dots, t_k]]$ (which means that $g(0, 0, \dots, 0) \neq 0$) such that $f \cdot g = t_k^n + a_{k-1}t_k^{n-1} + \cdots + a_1t_k + a_0$ for $a_i \in \Bbbk[[t_1, \dots, t_{k-1}]]$.

This theorem reduces factorization in $\Bbbk[[t_1, \dots, t_k]]$ to factorization in $\Bbbk[[t_1, \dots, t_{k-1}]][t]$ (the assumption is not too restrictive – make a linear change of variables). By induction hypothesis and Gauss's Lemma, $\Bbbk[[t_1, \dots, t_{k-1}]]$ is a UFD, and so is $\Bbbk[[t_1, \dots, t_{k-1}]][t_k]$.

Step 2: $\hat{\mathcal{O}}$ is a UFD implies $\mathcal{O}$ is a UFD. We need to show that $f, g \in \mathcal{O}$ (here $\mathcal{O}$ must be a local ring):

(1) $f|g$ is true in $\mathcal{O}$ if and only if $f|g$ in $\hat{\mathcal{O}}$
(2) f and g have no common divisors in $\mathcal{O}$ if and only if f and g have no common divisors in $\hat{\mathcal{O}}$.

Note that f can be irreducible in $\mathcal{O}$ and reducible in $\hat{\mathcal{O}}$. For example, $y^2 - (x^3 + x^2)$ is irreducible in $\mathcal{O}_{0,\mathbb{A}^2}$ but not in the completion.

So we know that $\bigcap_d \mathfrak{m}^d = 0 \subset \mathcal{O}$. Also know this module any $I \subset \mathcal{O}$, i.e. $\bigcap_d(I + \mathfrak{m}^d) = I$, which implies that $I \cdot \hat{\mathcal{O}} \cap \mathcal{O} = I$.

Then we can easily prove the first claim: $f|g \in \hat{\mathcal{O}}$, then $g \in f \cdot \hat{\mathcal{O}} \cap \mathcal{O} = f \cdot \mathcal{O}$ so $f|g$ in $\mathcal{O}$.

For the second claim, suppose that f and g have a non-unit greatest common factor $\alpha \in \hat{\mathcal{O}}$. Set $\beta = \frac{f}{\alpha}, \gamma = \frac{g}{\alpha} \in \hat{\mathcal{O}}$. Now $g \cdot \beta = f \cdot \gamma \in f\hat{\mathcal{O}}$. Take $n >> 0$ such that $\beta, \gamma \notin \mathfrak{m}^n\hat{\mathcal{O}}$. Take $b = \beta \bmod \mathfrak{m}^n\hat{\mathcal{O}}$.

Consider $gb = g\beta \mod g \cdot \mathfrak{m}^n\hat{\mathcal{O}}$. So $gb \in (f \cdot \hat{\mathcal{O}} + (g \cdot \mathfrak{m}^n)\hat{\mathcal{O}}) \cap \mathcal{O}$. Then $gb \in f \cdot \mathcal{O} + g \cdot \mathfrak{m}^n$.

So there is $b' = b \mod \mathfrak{m}^n$ such that $gb' \in f \cdot \mathcal{O}$. We know that $b' = \beta \mod \mathfrak{m}^n\hat{\mathcal{O}}$. Claim: $\beta|b'$.

This is true because $f|gb'$, so $\alpha\beta|\alpha\gamma b'$. Since α is the greatest common factor, we get $\beta|\gamma b'$. But β and γ have no common factors so $\beta|b'$. This implies that $b' = \beta$·unit. If $b' = \beta \cdot \phi$, then modulo $\mathfrak{m}^n\hat{\mathcal{O}}$, we have $b' = \beta$ and $\phi = 1$+zero-divisor.

Then replace β with b' and α with $\frac{f}{b'}$ (which is in $\mathcal{O}$ by first claim). □

This is cool because classification of hypersurfaces in affine space relied on unique factorization of the coordinate ring. Hence we will get that hypersurfaces are locally given by a single equation around a smooth point of $x \in X$ (stronger: its ideal is given by that equation).

21. November 13

Last time:

Theorem. *Suppose $x \in X$ is smooth. Then $\mathcal{O}_{x,X}$ is a UFD.*

21.1. **Hypersurfaces.** Note that: $f \in \mathcal{O}_{x,X}$ is irreducible if and only if $\langle f \rangle \subset \mathcal{O}_{x,X}$ is prime.

Consider $Y = V(f) \subset X$ (this is well-defined only up to passing to any neighborhood of X,). This happens if and only if Y is irreducible at x (i.e. x lies on a single irreducible component of Y). Also, $\text{codim} Y = 1$.

Conversely, let Y be a hypersurface in X. Then Y is a locally complete intersection at smooth $x \in X$. This is the same as the proof for hypersurfaces in affine space: we just use unique factorization.

Definition. A codimension k subvariety Y is locally complete intersection (ideal-theoretic) *at* $x \in X$ if: the ideal $I_{Y_x} = \{f \in \mathcal{O}_{x,X} : f|_Y = 0\}$ can be generated by exactly k elements of $\mathcal{O}_{x,X}$ ($I = \langle f_1, \dots, f_k \rangle$). Equivalently, there is an open affine neighborhood U of y such that $I(Y \cap U) \subset \Bbbk[U]$ is generated by k functions.

Example (of something that is not l.c.i.)**.** The union of coordinate axes in $\mathbb{A}^3$ is not l.c.i. at the origin.

Proposition. Suppose $x \in X$ is smooth, and $Y \subset X$ is a subvariety with $x \in Y$ smooth. Then Y is a locally complete intersection at x.

Sketch of proof. Assume $x \in Y \subset X \subset \mathbb{A}^n$. Then $T_{x,Y} \subset T_{x,X}$. Recall that $T_{x,Y}$ is the zero locus of $df(x)$ for $f \in I(Y)$, so we can find functions $f_1, \dots, f_k \in I(Y)$ such that $df_i(x)$ form a basis in $T^{\perp}_{x_i,Y} \subset T^*_{x,X}$. Then $Y \subset V(f_1, \dots, f_k) \subset X$. So Y and $V(f_1, \dots, f_k)$ are two varieties smooth at x of the same dimension. They are irreducible at x and hence coincide (near x).

Why do $\langle f_1, \dots, f_k \rangle$ generate the ideal $I_{x,Y} \subset \mathcal{O}_{x,X}$? I.e. why is this ideal radical? This can be checked in the completion $\hat{\mathcal{O}}_{x,X} \cong \Bbbk[[t_1, \dots, t_m]]$. Under appropriate choice of isomorphism, $I_{x,Y} \cdot \hat{\mathcal{O}}_{x,X} \cong (t_1, \dots, t_k)$, the latter certainly being radical. □

21.2. **Factorization in $\mathcal{O}_{x,X}$.** What is factorization in $\mathcal{O}_{x,X}$ geometrically?

If $f \in \mathcal{O}_{x,X}$ can be written as $f = \prod f_i^{m_i}$ where f_i's are irreducible germs, i.e. local equations of irreducible hypersurfaces passing through this point. We can now interpret m_i as the order of vanishing of f on $Y_i = V(f_i)$ near x (but this is independent of choice of x).

Corollary. *If X is smooth, and $f : X - Z \to \mathbb{A}^1$ for Z a subvariety of codimension at least two, then f extends regularly all of X.*

Proof. Near $x \in X$, $f = \dfrac{g}{h}$ with $g, h \in \mathcal{O}_{x,X}$, and we can take g and h to be coprime. Then $V(g)$ and $V(h)$ are hypersurfaces with no common components passing through x. Then f is not regular on $V(h) - V(g)$ (since multiplicity makes sense independent of the choice of smooth point). But $V(h) - V(g)$ is dense in $V(h)$, and also $V(h) - V(g) \subset Z$. Then $V(h) \subset Z$, so $V(h)$ is empty since dimension of Z is too low. □

Example. Smoothness is important. Consider $V(x_1, x_2) \cup V(x_3, x_4) \subset \mathbb{A}^4$. Then take f to be different constants on the two planes minus the origin.

Corollary. *Suppose we have $f : X \overset{rational}{\longrightarrow} \mathbb{P}^n$ with smooth X. Then f is regular outside of codimension* 2.

Proof. Locally, $f = (f_0 : \dots : f_n)$ with $f_i \in \mathcal{O}_{x,X}$. We can assume these are without any common factors in $\mathcal{O}_{x,X}$. Then $V(f_0, \dots, f_n)$ has codimension at least 2 (since a hypersurface will give a common factor). □

21.3. **Birational vs. Biregular classification of varieties.** Dimension 1 (Curves).

Suppose that X, Y are two projective smooth curves. For smooth varieties, irreducible is the same as connected; assume they are irreducible/connected.

Now, any $f : X \overset{\text{rational}}{\longrightarrow} Y$ is regular and any birational map is hence biregular.

So smooth projective curves (which over $\mathbb{C}$ are Riemann surfaces) up to birational equivalence are the same as smooth projective curves up to isomorhpism. Restatement: for any finitely generated extension $K \supset \Bbbk$ of transendence degree 1, there is at most one (up to isomorphism) smooth projective X such that $\Bbbk(X) \cong K$.

Fact: such X exists. It is easy to find an affine curve $Y \subset \mathbb{A}^n$ with $\Bbbk(Y) = K$. Then take $\bar{Y} \subset \mathbb{P}^n$, and again $\Bbbk(Y) \cong K$ (In fact, we can always take $n = 2$). The difficult part is making sure we get a smooth curve, i.e. how do we "resolve the singularities", i.e. starting with a singular projective $\bar{Y}$, how to we construct smooth projective X with $\nu : X \to \bar{Y}$ regular and birational?

This is tricky in general, but is not hard for curves:

(1) $\bar{Y} \subset \mathbb{P}^2$, we can blow up $\mathbb{P}^2$
(2) $\nu\colon X \to \bar{Y}$: near each singular $x \in Y$, each local ring is not a DVR, but is of dimension 1, so the problem can be thought of as it being not integrally closed. Thus, "normalization".

Example. We had a map $\mathbb{P}^1 \to V(y^2 - x^3)$. There is also a map $\mathbb{P}^1 \to V(y^2 - (x^3 + x^2)$, the inverse map is projection from the singular point.

22. November 15

Last time: we claimed that a map $f\colon X \overset{\text{rational}}{\longrightarrow} \mathbb{P}^n$ from smooth X was regular outside of a codimension 2. This applies to the classification problem of biregular versus birational equivalence: in dimension 1 (curves), any curve X' has a unique smooth projective model (up to isomorphism) X (i.e. X is birational to X').

Remark. For a smooth projective curve X, there is a bijection between $\{$points $x \in X\}$ and $\{$DVR's $\mathcal{O}_{x,X} \subset \Bbbk(X)$ such that $\mathcal{O}_{x,X} \supset \Bbbk\}$.

This reconstructs X as a ringed space from $\Bbbk(X)$.

Remark. Over $\mathbb{C}$, the classification of smooth projective curves is the same as the classification of compact Riemann surfaces [GAGA].

GAGA in general:

(1) An analytic subset of $\mathbb{P}^n$ is algebraic.
(2) Any map between analytic subsets of $\mathbb{P}^n$ is algebraic.

So the category of projective algebraic varieties over $\mathbb{C}$ is a full subcategory of the category of analytic varieties over $\mathbb{C}$ (those that embed into $\mathbb{P}^n$).

In fact: compact analytic manifolds do not necessarily have a lot of rational (meromorphic) functions, but smooth algebraic varieties do (as many algebraically independent ones as as the dimension+1).

22.1. Surfaces.

Example. Birational but not biregular smooth projecutve surfaces:

(1) $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. They are birational since they both contain $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$. They are not biregular since any two curves in $\mathbb{P}^2$ meet, but two curves in $\mathbb{P}^1 \times \mathbb{P}^1$ can not intersect. More explicitly, $\mathbb{P}^2-$curve is affine, but $\mathbb{P}^1 \times \mathbb{P}^1 - \{\infty\} \times \mathbb{P}^1 = \mathbb{A}^1 \times \mathbb{P}^1$ is neither affine nor projective.

Let us look more closely at the birationl map between $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. We have $(x\colon y\colon z) \to ((x\colon y), (x\colon z))$. This is regular on $\mathbb{P}^2 - \{(0\colon 1\colon 0), (0\colon 0\colon 1)\} \to \mathbb{P}^1 \times \mathbb{P}^1$.

The inverse map is $((x_0\colon x_1), (y_0\colon y_1)) \to (x_0y_0\colon x_1y_0\colon x_0y_1))$, which is regular on $\mathbb{P}^1 \times \mathbb{P}^1 - \{((0\colon 1), (0\colon 1))\}$. Notice this is like the Segre embedding with one component missing, so we are taking $\mathbb{P}^1 \times \mathbb{P}^1 \overset{\text{Segre}}{\to} \mathbb{P}^3$ projected on $\mathbb{P}^2$.

Note that the map from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$ contracts the entire line $(0\colon *\colon *)$ to $((0\colon 1), (0\colon 1))$. Similarly, the map from $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ contracts the lines $\{(0\colon 1)\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{(0\colon 1)\}$ through that point contract to two points on the line.

So the rational map $\mathbb{P}^2 \overset{\text{rational}}{\longrightarrow} \mathbb{P}^1 \times \mathbb{P}^1$ blows up the two points and contracts (blows down) the line through them.

22.2. Blow-up of affine space.

What is a blow-up (σ-process, monoidal transform).

Example (Key example)**.** The blow-up of $\mathbb{A}^2$ at 0.

Consider $\mathbb{A}^2 \overset{\text{rational}}{\longrightarrow} \mathbb{P}^1$ which sends $(x, y) \to (x\colon y)$. Take the *graph* of this (rational) map: by definition the closure of the graph of its restriction to $\mathbb{A}^2 - \{0\}$.

So we start with $\Gamma \subset (\mathbb{A}^2 - \{0\}) \times \mathbb{P}^1$ which is closed and is isomorphic to $\mathbb{A}^2 - \{0\}$ by the vertical line test (π_1).

Define $X = \bar{\Gamma} \subset \mathbb{A}^2 - \mathbb{P}^1$. Then $X = \Gamma \cup \{0\} \times \mathbb{P}^1$. One can say this is a graph of a multi-valued function. This is a variety, and it has a projection onto $\mathbb{A}^2$ (see the cover of Shafarevich).

This is called the $B|_0\mathbb{A}^2$, the blow-up centered at 0, and $C = \{0\} \times \mathbb{P}^1 \subset X$ is called the exceptional curve.

In fact we have $C =$lines in $T_0\mathbb{A}^2$.

The blow-up map σ is regular, birationl and is an isomorphism $\sigma\colon X - C \to \mathbb{A}^2 - \{0\}$.

Another description of X: pairs $\{$(point in $\mathbb{A}^2$, line through 0 and this point)$\}$.

In coordinates: $\mathbb{A}^2 \times \mathbb{P}^1 = \{((x,y),(z_0\colon z_1))\}$. Then X is given by $z_1 x = z_0 y$. On one of the charts $\mathbb{A}^1 \subset \mathbb{P}^1$, use $z = \frac{z_1}{z_0}$ as a non-homogenenous coordinate, and so inside the affine chart $\mathbb{A}^2 \times \mathbb{A}^1 = \mathbb{A}^3$, we have X is given by $zx = y$.

So $\mathbb{A}^3 \cap X \cong \mathbb{A}^2$ with coordinates (x,z) and $\sigma|_{X\cap\mathbb{A}^3}\colon \mathbb{A}^2 \to \mathbb{A}^2$ sends $(x,z) \to (x, xz)$. In particular, we see that X is smooth.

Versions of this: Take $x \in \mathbb{P}^2$ and $B|_x\mathbb{P}^2$ is the graph of projection $\mathbb{P}^2 \xrightarrow{\text{rational}} \mathbb{P}^1$.

Similarly, we can define $B|_x\mathbb{P}^n$ or $B|_x\mathbb{A}^n$, and in both cases we get a smooth variety, with exceptional locus is going to be $\mathbb{P}^{n-1}$. Note that $B|_x\mathbb{P}^n$ is projective.

22.3. Blow-up in general.

Definition. Suppose X is either affine or projective, take $x \in X$. Then define $B|_xX = \tilde{X}$ by embedding $X \hookrightarrow \mathbb{A}^n$ or $\mathbb{P}^n$ and taking its blow-up:

If $X \subset \mathbb{A}^n$ with $x = 0$. Then we have a blow-up $B|_0\mathbb{A}^n \to \mathbb{A}^n$ which is an isomorphism over $\mathbb{A}^n - 0$. So set $\tilde{X} = \sigma^{-1}(\bar{X} - \{0\})$ =graph of projection $X \xrightarrow{\text{rational}} \mathbb{P}^{n-1}$.

Example. Let X =nodal cubic, zero-locus of $y^2 - (x^3 - x^2))$, with characteristic not 2.

In coordinates, on one of the charts of the blow-up $\mathbb{A}^3 \subset \mathbb{A}^2 \times \mathbb{P}^1$ with coordinates (x,y,z) we have $B_0\mathbb{A}^2 \cap \mathbb{A}^3 \cong \mathbb{A}^2$ with coordiantes (x,z). So now $\sigma^{-1}(X) \cap \mathbb{A}^3$ is given by plugging in $y = xz$ in the defining equation, i.e. $V((xz)^2 - (x^3 + x^2)) = V(x^2(z^2 - x - 1))$.

$\sigma^{-1}(X)$ has two components: C, the exception curve, and the proper preimage $\tilde{X}$, which is give on one chart by $z^2 = x + 1$. This new curve is simpler in the sense that it is smooth.

In fact, any curve can be desingularized by successive blow-ups at the singular points.

23. November 20

Question (from student): why would we want to desingularize curves? Desingularizing the cuspidal cubic, we get $\mathcal{O}_{x,X} \to \mathcal{O}_{\tilde{x},\tilde{X}}$. The latter after completing might look like $\Bbbk[[t]]$, while the the completion of the former ring may look like $\{f \in \Bbbk[[t]]\colon f'(0) = 0\}$.

Example. For the cuspidal case, the ring would look like $\{(f_1,f_2)|f_1(0) = f_2(0)\} \subset \Bbbk[[t_1]] \times \Bbbk[[t_2]]$...

Remark. Embedded resolution of singularities: consider a curver $X \subset \mathbb{A}^2$ singular at 0; blowing up at 0 to obtain $B|_0\mathbb{A}^2$, we will get a blow-up of the curve tagnent to the exceptional curve. This simplifies the picture, even if there are more curves involved.

This is trying to resolve "a divisor with normal cosets".

23.1. Blow-ups in general. Last time: we defined precisely the blow-up of $\mathbb{A}^n$ at 0 as the graph of $\mathbb{A}^n \xrightarrow{\text{rational}} \mathbb{P}^{n-1}$ which is regular outisde the origin, so $B|_0\mathbb{A}^n \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$. We have a map $\pi_1 = \sigma\colon B|_0\mathbb{A}^n$, which is bjiective outside 0, while $\sigma^{-1}(0) \cong \mathbb{P}^{n-1}$.

The real picture is using spherial coordinates: we have $\mathbb{R}^n \leftarrow S^{n-1} \times \mathbb{R}^{\geq 0}$. In the complex world this cannot be done with a sphere; it must be done with projective space.

For $0 \in X \subset \mathbb{A}^n$, we set $B|_0X = \sigma^{-1}(\bar{X} - \{0\})$ =proper preimage of X under σ. By definition this is contained in $B|_0\mathbb{A}^n$.

Equivalently: $B|_0X$ =graph of $X \xrightarrow{\text{rational}} P^{n-1}$ where $\mathbb{P}^{n-1}$ is givn by $(z_1, z_2, \ldots, z_n)$ where z_i are the coordinates on $\mathbb{A}^n \supset X$. It is only important that the common zero locus of z_i is the 0: result will be the same for any collection of functions satisfying that property.

More explcitly: assume X is affine, with $0 \in X$. Then take any set $f_1, \ldots, f_n \in \Bbbk[X]$ such that $I_0 = \{f_1, \ldots, f_n\}$. Then $B|_0X$ =graph of $X \xrightarrow{\text{rational}} \mathbb{P}^{n-1}$ given by $f = (f_1\colon \ldots\colon f_n)$. By construction this lives insode $X \times \mathbb{P}^{n-1}$.

Exercise. This does not depend (up to isomorphism) on the choice of f_i's.

"Now I think I actually talked myself into a corner." (regarding an error earlier)

Remark (Generalization)**.** Suppose $f_1, \ldots, f_n \in \Bbbk[X]$ are such that $I_Y = (f_1, \ldots, f_n)$. Then the graph of $(f_1\colon \ldots\colon f_n)$ is $B|_YX$ =blow-up along Y.

Definition. We can now define $B|_X$ for any variety X at $0 \in X$ by taking an affine $U \ni x$, so that $X = U \cup (X - \{0\})$ glued along $U - \{0\}$, then $B|_0 X = B|_0 U \cup (X - \{0\})$ glued along $U - \{0\}$.

Similarly, given $X \supset Y$ where Y is closed and X is an arbitrary variety, then we can define $B|_Y X$ as the gluing of $B|_{Y \cap U} U$ for affine charts $U \subset X$.

Proposition (Main Properties of Blow-ups)**.**

(1) Blow-up of a projective variety is projective.
(2) Blow-up of a smooth variety along a smooth locus is smooth.

Remark (Analaytically)**.** The blow-up of a smooth X along a smooth Y looks as follows: $\sigma^{-1}(Y)$ locally looks like $Y \times \mathbb{P}^{k-1}$ where $k = \text{codim} Y$. In fact, $\sigma^{-1}(y) \cong \mathbb{P}^{k-1}$-projective space of normal directions to Y (quotient of tangent space of X by tangent space of Y).

23.2. **Applications of blow-up.** We can resolve singularities.

Theorem (Hironaka's Theorem)**.** *In characterstic* 0*, singular projective* X *can be desingularized by a sequence of blow-ups (not just at points).*

This is a hard theorem.

Back to the difference between birational and biregular classification. (Sufrace, dim = 2). Any surface (by Hironaka's theorem) has a smooth projective model, but how many modeles are there?

Fact: Any birational map between smooth projective surfaces is a sequence of blow-ups [at single points] and blow-downs).

Example. $\mathbb{P}^2 \overset{\text{rational}}{\longrightarrow} \mathbb{P}^1 \times \mathbb{P}^1$ is a composition of blow-ups at two points and contraction(blow-down) of line joining them.

Remark. The fact is more useful once we know which curves can be contracted. The answer is Castelnuovo's criterion: A curve C on a smooth (projective) surface S can be contracted if and only if $C \cong \mathbb{P}^1$, and its self-intersection is -1. E.g. C is rigid: unique curve in its cohomological class $H^2(S, \mathbb{Z})$ (analytically)...

Definition. A *minimal model* of surface is a model with no exceptional curves.

A minimal model is unique up to isomorphism except for certain speical (well-understood) cases (surfaces isomorphic to $\mathbb{P}^1 \times$curve).

In higher codimension: take X a 3-fold, then we can have a flop between two models X' and X'' that are blow-downs of X. If we blow-down both of those we can get a minimal singular model.

24. November 27

New stuff!

24.1. **Divisors.** Let X be a smooth variety (smoothness is a bit of an overkill; normal varieties will suffice).

Recall that if X is smooth, then for any point $x \in X$, $\mathcal{O}_{x,X}$ is a UFD. So if $f \in \mathcal{O}_{x,X} - \{0\}$, then we can write it $f = \prod g_i^{n_i}$ where $g_i \in \mathcal{O}_{x,X}$ are primes and $n_i \geq 0$, so locally $V(g_i) = Z_i$ is an irreducible hypersurface passing through this point. The geometric meaning is that n_i is the order of zero of f along Z_i.

If $f \in \Bbbk(X) - \{0\}$, we can do the same thing, but now the n_i can be negative, so if $n_i < 0$ we say that f has a pole or order $-n_i$ along Z_i.

Definition. A *prime divisor* on X is a codimension 1 irreducible closed subvariety.

We can restate all this so it does not depend on the particular point.

Definition. A *local equation* of a prime divisor Z is $g \in \Bbbk[U]$ for some open affine $U \subset X$ such taht $U \cap Z \neq \emptyset$ and $\langle g \rangle = I(Z \cap U) \subset \Bbbk[U]$.

If $g \in \Bbbk[U]$ and $g' \in \Bbbk[U']$ are two local equations for Z (on different opens, possibly), then we can take affine open $U'' \subset U \cap U'$ such that $U'' \cap Z \neq \emptyset$; then for $g|_{U''}$ and $g'|_{U'}$, $g = g'h$ for a regular $h \colon U'' \to \mathbb{A}^1 - \{0\}$.

Note: if X is separated (actually a variety), then $U \cap U'$ is automatically affine.

Definition. Suppose $f \in \Bbbk(X)$ is a rational function (assume X is irreducible for simplicity's sake; since X is smooth its connected components are the irreducible pieces). Then f has neither zero nor pole along a prime divisor Z_i if both f and f^{-1} are regular somewhere on Z.

Otherwise, $\operatorname{ord}_z f = n$ means $\dfrac{f}{g^n}$ has neither zeros or poles along Z (where g is a local equation for Z). We say that $n =$*order of zero of* f *along* Z, and $-n =$*order of pole of* f *along* Z. The choice of g as a local equation of Z does not matter.

Finally, $\operatorname{ord}_z 0 = +\infty$.

Algebraically, $\mathcal{O}_{Z,X} = \{f \in \Bbbk(X) | f \text{ is regular somewhere on } Z\}$. Then $\operatorname{ord}_z f = 0$ if and only if f has neither zero nor pole, i.e. $f \in \mathcal{O}_{Z,X}$ is a unit. Now, $\mathcal{O}_{Z,X}$ is local, so its maximal ideal $\mathfrak{m}$ consists of functions that are $\{f \colon f|_Z = 0 \in \Bbbk(Z)\}$. This is true always.

For the special case where Z is a hypersurface in a smooth variety X, then $g \in \Bbbk(X)$ is a local equation of Z if and only if $g \in \mathcal{O}_{Z,X}$ and $\langle g \rangle = \mathfrak{m}$. Hence, $\mathcal{O}_{Z,X}$ is a DVR, and ord_Z is the valuation.

Example. If $X =$smooth curve, then $Z =$point, and the local ring of a point in a smooth curve is a DVR.

Remark. If $X =$affine, then $I(Z) \subset \Bbbk[X]$ is a prime ideal, and $\mathcal{O}_{Z,X}$ is then exactly the localization $\Bbbk[X]_{I(Z)}$.

Proposition. The order satisfies the following properties.

(1) $\operatorname{ord}_Z(f_1 \cdot f_2) = \operatorname{ord}_Z f_1 + \operatorname{ord}_Z f_2$
(2) $\operatorname{ord}_Z(f_1 + f_2) \geq \min\{\operatorname{ord}_Z f_1, \operatorname{ord}_Z f_2\}$

Fix $f \in \Bbbk(X) - \{0\}$. Then f is regular on some $U \subset X$. We could have prime divisors in the complement of U. But for any prime divisor Z such that $Z \cap U \neq \emptyset$, $\operatorname{ord}_Z f \geq 0$. Dually, f^{-1} will be regular on some $V \subset X$, and for any Z such that $Z \cap V \neq \emptyset$, $\operatorname{ord}_Z f \leq 0$.

Hence: $\operatorname{ord}_Z f \neq 0$ for all but finitely many Z (there are only finitely many Z outside of $U \cap V$?).

Definition. A *divisor* on X is an element of the free abelian group $\operatorname{Div} X = \bigoplus_{Z \subset X} \mathbb{Z} \cdot Z$.

Example. When $X =$curve, $D = \sum n_i x_i$ is a finite sum of points $x_i \in X$.

Definition. Given $f \in \Bbbk(X) - \{0\}$, its divisor is $(f) = \sum(\operatorname{ord}_Z f) \cdot Z \in \operatorname{Div} X$. In fact, $f \to (f)$ is a group homomorphism $\Bbbk(X)^\times \to \operatorname{Div} X$.

By definition, $D = \sum n_i Z_i \geq 0$ if all $n_i \geq 0$, in which case we say D is effective.

Example. Suppose $f \in \Bbbk(X)^\times$, and $(f) \geq 0$. We want to show that if f has no poles along any prime divisor, then $f \colon X \to \mathbb{A}^1$ is regular.

This is so because the complement of the domain of regularity of f has to have codimension 2, hence f can be extended because X is smooth.

Corollary. *$(f) = 0$ implies $(f) \geq 0$ and $-(f) \geq 0$, but then $-(f) = (f^{-1}) \geq 0$, so both f and f^{-1} are regular on all of X, so $f \colon X \to \mathbb{A}^1 - \{0\}$.*

Definition. For $f \in \Bbbk(X)^\times$, we can let $(f)_0 = \sum_{\operatorname{ord}_Z f \geq 0} \operatorname{ord}_Z f Z$ and $(f)_\infty = \sum_{\operatorname{ord}_Z f \leq 0} (-\operatorname{ord}_Z f) Z \geq 0$, so $(f) = (f)_0 - (f)_\infty$. $(f)_0$ is *the divisor of zeroes* and $(f)_\infty$ *is the divisor of poles.*

Lemma. *The domain of regularity of f is precisely $X - \bigcup U_{Z \subset X, \operatorname{ord}_Z f < 0} Z$.*

Remark. We needed two things about X:

(1) Any prime divisor has a local equation
(2) The ability to extend regular functions across codimension 2 subsets of any open in X.

These hold on normal varieties (the geometric counterpart of integrally closed ring; this includes some singular varieties).

But the theory with normal varieties have some displeasing features:

If X is singular, then this notion of divisors (Weil divisors) is in some ways not as good as another notion (Cartier divisors).

The idea behind Cartier divisors: we give a notion of what it means for two functions to locally have the same kind of singularities. Local singularities are going to be recorded as $(U_i \subset X, f_i \in \Bbbk(U_i)^\times)$. Concretely, we take a family $(U_i, f_i \in \Bbbk(U_i)^\times)$ with $\bigcup U_i = X$ and $\dfrac{f_i}{f_j}$ has neither zeroes nor poles on $U_i \cap U_j$.

This is an explcit way to describe global sections of the sheaf $\Bbbk(X)^\times / \mathcal{O}_X^\times$.

25. November 29

Last time: we talked about divisors. If X is smooth, then $\operatorname{Div} X = \{\sum n_i Z_i : n_i \in \mathbb{Z}, Z_i \subset X$ irreducible closed hypersurface$\}$.

We have $f \to (f) = \sum(\operatorname{ord}_Z f)Z$, which is a group homomorphism $\Bbbk(X)^\times \to \operatorname{Div} X$.

Example. $(f) = 0$ if and only if $f \colon X \to \mathbb{A}^1 - \{0\}$, i.e. $f \in \Gamma(X, \mathcal{O})^\times$ (ring of globally defined functions, its units in particular)

There are two classes of questions:

- "Multiplicative": Is there a function f with exactly these zeroes/poles?
- "Additive": Is there f with zeroes/poles no worse than given.

25.1. Multiplicative question.

Definition. A divisor is *principal* if $D = (f)$.

We let $\operatorname{Pic} X = \operatorname{Div} X / \{(f) \colon f \in \Bbbk(X)^\times\}$ be the *divisor class group*/Picard group; two divisors with $[D_1] = [D_2]$ (if and only if $D_1 - D_2 = (f)$), we say are *rationally equivalent*.

Example.

(1) $X = \mathbb{A}^1$. Then $\operatorname{Div} X \ni \sum n_i x_i$, with $x_i \in \mathbb{A}^1$. All of these are principal ($\prod(x - x_i)^{n_i}$), so $\operatorname{Pic} \mathbb{A}^1 = 0$

(2) $X = \mathbb{A}^n$, then $\operatorname{Div} X = \sum n_i Z_i$. Each $Z_i = (f_i)$ where $\langle f_i \rangle = I(Z_i)$. Principal divisors form a subgroup and contain the generators, so again $\operatorname{Pic} \mathbb{A}^n = 0$.

(3) $U = \mathbb{A}^1 - \{0\}$. Again, $\operatorname{Pic} \mathbb{A}^1 - \{0\} = 0$ since $\operatorname{Pic} \mathbb{A}^1 = 0$.

This is one way. More formally: we have a surjective restriction map $\operatorname{Div} X \to \operatorname{Div} U$ (a hypersurface in X intersects to get a hypersurface in U). The kernel is precisely divisors in $X - U$. It sends principal divisors to principal divisors, so induces a surjection $\operatorname{Pic} X \to \operatorname{Pic} U$, so $\operatorname{Pic} U = 0$.

A second way: the only way we needed to prove that $\operatorname{Pic} \mathbb{A}^n = 0$ was unique factorization in $\Bbbk[\mathbb{A}^n]$. So $\Bbbk[U] = \Bbbk[x, x^{-1}]$ which is also a UFD.

Proposition. If X is affine (and smooth), then $\Bbbk[X]$ a UFD implies $\operatorname{Pic} X = 0$. In fact, this is an if and only if.

Proof. Exercise. □

Example. Example continued.

(4) $\operatorname{Pic}(\mathbb{A}^2 - \{0\} = 0$ by being the quotient of $\operatorname{Pic}(\mathbb{A}^2)$. Generally, if $U \subset X$ is open and codimension $X - U$ is at least 2, then the restriction map is an isomorphism because no divisors can hide in the complement of U. So $\operatorname{Div} X \overset{\cong}{\to} \operatorname{Div} U$ and induces $\operatorname{Pic} X \overset{\cong}{\to} \operatorname{Pic} U$, as D on X is princial if and only if $D|_U$ is principal.

Another way to say this is that $\Bbbk(X)^\times \to \operatorname{Div} X \to \operatorname{Pic} X \to 0$ and $\Bbbk(X)^\times \to \operatorname{Div} U \to \operatorname{Pic} U \to 0$, and the restriction map which is an isomorphism of the first two, must be then an isomorphism of the third.

Example. $Z_i = V(f_i)$ for unique irreducible homogeneous f_i. We want to take $\prod f_i^{n_i}$, which makes sense as a ratioanl function if (and only if) $\sum n_i \deg(f_i) = 0$. So this is the only obstruction to a divisor being principal.

Remark (Notation)**.** $\deg Z_i = \deg f_i$ for $I(Z_i) = \langle f_i \rangle$. Then $\deg \sum n_i Z_i = \sum n_i \deg Z_i$. Then we have $\operatorname{Div} \mathbb{P}^n \overset{\deg}{\to} \mathbb{Z}$, and its kernel is precisely all principal divisors, so in fact $\operatorname{Pic} \mathbb{P}^n \cong \mathbb{Z}$. degree 0.

Anoter way: we can find a rational function with prescribed zeroes and poles on $\mathbb{A}^n \subset \mathbb{P}^n$. Then the complement is precisely one hypersurface, and on it the poles and zeroes either match or they do not. The difference is exactly the degree of the divisor.

Topologically: if X is projective and smooth over $\mathbb{C}$, then $Z \subset X$ is closed of real dimension $2n - 2$ if $\dim X = n$. So it defines $[Z] \in H_{2n-2}(X, \mathbb{Z})$. Can use Poincare duality to consider $[Z] \in H^2(X, \mathbb{Z})$ (the latter will be defined even if X were not projective). Then we get a map $\operatorname{Div} X \to H^2(X, \mathbb{Z})$.

Claim: $\operatorname{Div} X \to H^2(X, \mathbb{Z})$ factors through $\operatorname{Pic} X$, inducing $c_1 \colon \operatorname{Pic} X \to H^2(X, \mathbb{Z})$. Application: $\operatorname{Pic} \mathbb{P}^n \overset{c_1}{\to} H^2(\mathbb{P}^n, \mathbb{Z})$, both are $\mathbb{Z}$. "Proof" of claim: Suppose $f \in \Bbbk(X)^\times - \Bbbk$. Then we want to consider $f \colon X \to \mathbb{P}^1$, but this is not regular everywhere, but it is regular outside of codimension 2, so we consider $: X - (\text{codim}2) \to \mathbb{P}^1$. Then $[(f)_0] = f^*([0])$ where $[0] \in H^2(\mathbb{P}^1, \mathbb{Z})$. Similarly, $[(f)_1] = f^*([\infty])$ with $[\infty] \in H^2(\mathbb{P}^1, \mathbb{Z})$. Hence, $[(f)] = 0$ as the difference of the two points vanishes.

25.2. **Local-to-global approach to divisors.** Note: locally, every divisor is principal: given $D \in \operatorname{Div} X$, there is an $X = \bigcup_i U_i$ such that $D|_{U_i} = (f_i)$ for $f_i \in \Bbbk(U_i)^\times$. This relies on UFD in $\mathcal{O}_{x,X}$, which fails for general normal varieties.

Conversely, given $X = \bigcup U_i$ and $f_i \in \Bbbk(U_i)^\times$, we can reconstruct D, provided that $(f_i) \in \operatorname{Div} U_i$ and $(f_j) \in \operatorname{Div} U_j$ agree on $U_i \cap U_j$ (i.e. Divisor groups $\operatorname{Div}(U)$ for $U \subset X$ form a sheaf).

Explcitly, we need $\dfrac{f_i}{f_j} = g_{ij} \in \Gamma(U_i \cap U_j; \mathcal{O})^\times$.

D is principal if there exist ϕ_i's with $\phi_i \in \Gamma(U_i, \mathcal{O})^\times$ such that $\dfrac{f_i}{\phi_i} = \dfrac{f_j}{\phi_j}$, this is if and only if $g_{ij} = \dfrac{\phi_i}{\phi_j}$.

But each (g_{ij}) is a 1-Cech cocycle, and the divisor will be principle if and only if (g_{ij}) is a coboundary.

Theorem. $\operatorname{Div} X \to H^1(X, \mathcal{O}^\times)$. *This induces an isomorphism from* $\operatorname{Pic} X$ *to* $H^1(X, \mathcal{O}^\times)$.

Sketch of proof. The map is well-defined. If we change f_i's to $f_i\psi_i$, then the Cech cocylce does not change.

The map is bijective since given g_{ij} we can construct f_i by taking $f_1 = 1$ on some open set, and then $f_i = g_{i1} f_1$. ☐

In a fancier way $1 \to \mathcal{O}^\times \to \mathcal{K}^\times$ is a map fo invertible sheaf of regular functions to the sheaf of invertiable rational functions=constant sheaf. Easy exercise: in Zariski topology, Cech cohomology of a constant sheaf is trivial.

We can complete to a short exact sequence $1 \to \mathcal{O}^\times \to \mathcal{K}^\times \to \mathcal{K}^\times/\mathcal{O}^\times \to 1$, gives us a long exact sequence in cohomology $\Gamma(X, \mathcal{K}^\times) \to \Gamma(X, \mathcal{K}^\times/\mathcal{O}^\times) \to H^1(X, \mathcal{O}^\times) \to H^1(X, \mathcal{K}^\times) = 0$ and this is exactly $\Bbbk(X)^\times \to \operatorname{Div} X \to \operatorname{Pic} X \to 0$.

26. December 4

Last time: X =smooth irreducible, we defined $\operatorname{Pic} X = \operatorname{Div} X/\{(f) \colon f \in \Bbbk(X)\}$. Then $\operatorname{Pic} X \cong H^1(X, \mathcal{O}_X^x)$ – first Cech cohomology on the sheaf of invertible regular functions.

Remark. Suppose X =smooth over $\mathbb{C}$. Last time, we talked about $c_1 \colon \operatorname{Pic} X \to H^2(X, \mathbb{Z})$. This can be explained as follows.

Consider the sheaf $\mathcal{O}_{X^{\text{an}}}$ =sheaf of holomorphic functions. Then $\mathcal{O}_{X^{\text{an}}}$ =sheaf of holomoprhic functions without zeroes.

There is a map $H^1(X, \mathcal{O}_X^x) \to H^1(X, \mathcal{O}_{X^{\text{an}}}^x)$. (GAGA states that this is an isomorphism if X is projecttive).

Analytically, consider $0 \to \mathbb{Z} \overset{2\pi i}{\to} \mathcal{O}_{X^{\text{an}}} \overset{\exp}{\to} \mathcal{O}_{X^{\text{an}}}^x \to 1$.

This gives

$$\begin{array}{ccc} H^1(X, \mathcal{O}_{X^{\text{an}}}^x) & \longrightarrow & H^2(X, \mathbb{Z}) \\ \uparrow & \nearrow_{c_1} & \\ H^1(X, \mathcal{O}_X^x) & & \end{array}$$

26.1. **Divisors vs ideals.** Suppose X =affine and $f \in \Bbbk[X] - \{0\}$. Then we have $0 \le (f) \in \operatorname{Div} X$ and $\langle f \rangle \subset \Bbbk[X]$. Both of these record the zeroes of f with multiplicity. This is not surprising as both are determined up to scalar by f.

Now, let us extend to effective divisors that are not necessarily principal. Given $0 \le D \in \operatorname{Div} X$, consider $D \mapsto I(D) = I \subset \Bbbk[X]$ given by $I = \{g \colon (g) \ge D\}$. If $f \in \Bbbk[X]$, then $(f) \mapsto \langle f \rangle = \Bbbk[X] \cdot f$. This boils down to the fact that f divides g if and only if $(f) \le (g)$.

Claim: this is a bijection between $\operatorname{Div}^{\ge 0} X = \{D \colon D \ge 0\}$ and locally principal ideals $I \subset \Bbbk[X]$.

Definition. I is locally principal if (either of the following equivalent conditions holds):

(1) For any $x \in X$ there is $f \in \Bbbk[X]$, $f(x) \ne 0$ such that $I \cdot \Bbbk[X]_f \subset \Bbbk[X]_f$ is principal.
(2) For any $x \in X$, $I \cdot \mathcal{O}_{x,X} \subset \mathcal{O}_{x,X}$ is principal.

The inverse map to $\operatorname{Div}^{\geq 0} X \to \{g\colon (g) \geq D\}$ can be constructed by $\inf\{(f)\colon f \in I\}$. Actually, we have a more general claim:

There is a bijection $\operatorname{Div} X \to$locally principal fractions ideals for $\Bbbk[X]$.

Definition. A *fractional ideal* is a finitely generated $\Bbbk[X]$-submodule in $\Bbbk(X)$, so $I = \left\langle \dfrac{f_1}{g_1}, \dots, \dfrac{f_n}{g_n} \right\rangle \subset \dfrac{1}{g_1 \cdots g_n}\Bbbk[X]$.

Remark.

(1) The advantage to fractional ideals is that we get an actual group. The group law in $\operatorname{Div} X$ corresponds to the product operation on fractional ideals.

(2) Suppose that D_1 and D_2 are rationally equivalent., i.e. $D_1 = D_2 + (f)$. Then $I(D_1) = I(D_2) \cdot \langle f \rangle = I(D_2) \cdot f$.

In particular, $I(D_1) \cong I(D_2)$ as $\Bbbk[X]$-modules. Conversely, any isomorphism of fractional ideals is of this form.

Proof.

$$\begin{array}{ccc} I_1 & \longrightarrow & I_2 \\ \downarrow & & \downarrow \\ \Bbbk(X) = I_1 \otimes \Bbbk(X) & \longrightarrow & I_2 \otimes \Bbbk(X) = \Bbbk(X) \end{array}$$

□

Theorem. $\operatorname{Pic} X =$*isomorphism classes of finitely generated* $\Bbbk[X]$*-modules that are locally free of rank* 1.

Sktech of proof. We only need to show that a rank 1 locally free finitely generated module M is isomorphic to a fractional ideal. The proof is simialr as before: $M \xrightarrow{i} M \otimes \Bbbk(X)$ and $M \otimes \Bbbk(X) \cong \Bbbk(X)$ because it is a vector space of dimension 1. We must check that the first map is injective, but this is just chasing localizations.

Concretely i is injective because it is injective locally. More precisely we have the following two equivalent formulations:

(1) Let R be a commutative ring, and $\phi\colon M \to N$ is a map of R-modules, and that $g_1, \dots, g_n \in R$ generate $\langle g_1, \dots, g_n \rangle = 1$. Consider $\phi_{g_i}\colon M_{g_i} \to N_{g_i}$. Then ϕ is injective if and only if ϕ_{g_i} is injective. ϕ

(2) Consider, for any prime ideal $\mathfrak{p} \subset R$, $\phi_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$. Then ϕ is injective if and only if $\phi_{\mathfrak{p}}$ is injective for all $\mathfrak{p}$.

(3) If R is a finitely generated algebra over a field (maybe just Noetherian), and M and N are finitely generated module, then ϕ is injective if and only if $\phi_{\mathfrak{m}}$ is injective for all maximal $\mathfrak{m}$.

□

Exercise. The group law in $\operatorname{Pic} X$ corresponds to the tensor product of modules.

Sometimes instead of "locally free of rank 1" some people say "invertible module" since a finitely generated module M is locally free of rank 1 if and only if there exists N such that $N \otimes M = \Bbbk[X]$.

Remark. If X is not affine, we have to consider shaves of (fractional) ideals, or sheaves of modules, over $\mathcal{O}_X$, i.e. sheaves of $\mathcal{O}_X$-modules.

26.2. **Divisors and line bundles.** There are bijections:

$$\begin{array}{ccc} \operatorname{Pic} X & \longrightarrow & H^1(X, \mathcal{O}_X^x) \\ & \swarrow & \\ & \text{Isomorphisms classes of line bundles on } X & \end{array}$$

Definition. An *algebraic vector bundle* E on a variety X is:

- A variety E equipped with a regular map $p\colon E \to X$

- The structure of a vector space over $\Bbbk$ on $p^{-1}(x)$ for all $x \in X$

such that there is a cover $X = \bigcup U_i$ and isomorphisms $p^{-1}(U_i) \cong U_i \times \mathbb{A}^n$ agreeing with all the data.

Explicitly: E is glued from $p^{-1}(U_i) = U_i \times \mathbb{A}^n$. For any i, j we have two maps $\beta_i \colon p^{-1}(U_i \cap U_j) \to (U_i \cap U_j) \times \mathbb{A}^n$ and $\beta_j \colon p^{-1}(U_i \cap U_j) \to (U_i \cap U_j) \times \mathbb{A}^n$.

Thus the transition map $\beta_j \circ \beta_i^{-1}$ has to be regular, has to agree with π_1, i.e. the map looks like $(x, v) \mapsto (x, \phi(v, x))$, and has to agree with the structure of a vector space, $\phi(v, c)$ has to be linear in v so it looks like an invertible $n \times n$ matrix with elements regular functions on X.

27. December 6

27.1. **(Algebraic) vector bundles.** Last time: an algebraic vector bundle is $p \colon E \to X$ such that p is regular, $p^{-1}(x)$ has the structure of a vector space so that E is locally trivial over X, i.e. locally it looks like

$$\begin{array}{ccc} U \times \mathbb{A}^n & \xrightarrow{\cong} & p^{-1}(U) \\ {\scriptstyle \pi_1}\downarrow & \swarrow_{p} & \\ U & & \end{array}$$

Here $n = \mathrm{rk}(E)$, is the rank of E.

Explicitly, if E over X is given by open sets $\bigcup U_i = X$ its rank n, and transition functions: invertible $n \times n$ matrices $\gamma_{ij} \in \mathrm{Gl}(n, \Gamma(U_i \cap U_j, \mathcal{O}))$ where $\Gamma(U_i \cap U_j, \mathcal{O})$ are regular functions on $U_i \cap U_j$. The compatibility requirement is $\gamma_{ij}\gamma_{jk} = \gamma_{ik}$ on $U_i \cap U_j \cap U_k$.

Example.

(0) Trivial bundle $X \times \mathbb{A}^n$

(1) Tautological vector bundle over $\mathrm{Gr}(k, n)$ given by $E = \{(v, V) \colon v \in V\}$. Here $\mathrm{rk}E = k$.

E.g. $E \to \mathbb{P}^n$, and $E =$Blow-up of $\mathbb{A}^{n+1}$.

(2) Take $X =$smooth. Then we have $E = TX =$tangent bundle which is $TX = \bigcup_{x \in X} T_x X$. We need to make TX into a variety. A system of local parameters give basis of this space over each point, and also around that point. So if t_i's are local parameters on U, then $p^{-1}(U) = TU \cong U \times \mathbb{A}^n$ with $(u, \xi) \mapsto (u, \langle \xi, dt_i(u) \rangle_{i=1,\dots,n})$.

$T^*X =$cotangent bunlde can be defined $\bigsqcup_{x \in X} T_x^* X$. This is a general construction: given $p \colon E \to X$, we can define its dual $p \colon E^* \to X$ as follows: $E^* = \bigsqcup_{x \in X} (E_x)^*$ where $E_x =$fiber. Locally, $E \cong \mathbb{A}^n \times U$, so $E^* \cong (\mathbb{A}^n)^* \times U$ (we should check this is indepdenent of the trivializations).

If E is given by $\gamma_{ij} \in \mathrm{GL}(n, \Gamma(U_i \cap U_j, \mathcal{O}))$, then the transition functions for E^* are going to be $(\gamma_{ij}^t)^{-1}$. Note that inversion is fine because we have $U_i \cap U_j \overset{\gamma_{ij}}{\to} \mathrm{GL}(n) \overset{A \mapsto A^{-1}}{\to} \mathrm{GL}(n)$ since $\mathrm{GL}(n)$ is $\Bbbk[\mathrm{GL}(n)] = \Bbbk[\text{entries}, \det^{-1}]$.

This also works for: $E_1 \to X$, $E_2 \to X$ give us $E_1 \oplus E_2$, $E_1 \otimes E_2$, $E_1^{\otimes k}$, $\Lambda^k E_1$, $\mathrm{Sym}^k E \dots$

E.g. E_k is given by $\gamma_{ij}^{(k)}$ for $k = 1, 2$. Then $E_1 \oplus E_2$ is going to be given by $\begin{pmatrix} \gamma_{ij}^{(1)} & 0 \\ 0 & \gamma_{ij}^{(2)} \end{pmatrix}$.

27.2. **Sections of a vector bundle.** Vector bundles form a category.

Definition. A *section* is a regular map $s \colon X \to E$ such that $p \circ s = \mathrm{id}$.

Example. If $E = \mathbb{A}^n \times X$, then $s =$n-tuple of regular functions. So $s(x) = E_x$ depending regularly on x.

Example.

(1) Sections of the tangent bundle TX are regular vector fields.

(2) Sections of the cotangent bunlde T^*X are (regular) differential 1-forms.

Sections of $\Lambda^k T^* X$ are (regular) differential k-forms.

(3) Consider

$$\begin{array}{ccc} U \times \mathbb{A}^n & \xrightarrow{\cong} & p^{-1}(U) \\ {\scriptstyle \pi_1}\downarrow & \swarrow_{p} & \\ U & & \end{array}$$

There are standard sections $U \to \mathbb{A}^n \times U$ given by $e_i \colon U \to (e_i, u)$. So a trivialization of $p^{-1}(U) \to U$ is the same (gives rise to) a *basis* of sections $e_1, \dots, e_n \colon U \to p^{-1}(U)$ that gives a basis in any fiber. Conversely, such a basis determines a trivialization $p^{-1}(U) \xrightarrow{\cong} \mathbb{A}^n \times U$.

What do sections on of a vector bundle on X form? Clearly (must check algebraic dependence, but that's easy) they form a vector space. Moreover, we can take linear comibnations of sections with coefficients in $\Gamma(X, \mathcal{O})$. This gives us a module $\Gamma(X, E) = \{\text{sections } s \colon X \to E\}$ over $\Gamma(X, \mathcal{O})$.

Example. If E is trivial, then $\Gamma(X, E)$ is a trivial module (of the same rank).

Assume that X is affine. Then:

1. For any vector bundle $E \to X$, $\Gamma(X, E)$ is a finitely generated module that is locally free (need to show: if $U = X - V(f)$, then $\Gamma(U, E) = \Gamma(X, E)_f = \Gamma(X, E) \otimes_{\Bbbk[X] \Bbbk[U]})$
2. This is an equivalence of categories (between vector bundles over X and finitely generated locally free $\Bbbk[X]$-modules).

Remark. If X is not affine, we still have the fact, but we have to work with sheavevs of sections, which are sheaves of modules over the structure sheaf of algebras.

27.3. **Line bundles and divisors.** Tkae X =smooth.

Theorem. *Line bundles up to isomorphism are in bijection with* $\operatorname{Pic} X$.

Concretely:

$$\begin{array}{ccc} \text{L. bundles} & \xrightarrow{\quad\cong\quad} & \operatorname{Pic} X \\ & {}_{\gamma_{ij}}\searrow \qquad \swarrow_{\cong} & \\ & H^1(X, \mathcal{O}_X^x) & \end{array}$$

The transition functions γ_{ij} for a line bundle $\mathcal{L}$ form a 1-cocycle.

Another formula for correspondence: line bundles$\to$ $\operatorname{Pic} X$. Suppose we have $p \colon L \to X$. Locally, over $U \subset X$ we have L is trivial, and hence we can choose a $s \colon U \to L$ with no zeroes. This is a "rational section" of L over X. To such s we associate its divisor (s) (this is a local construction).

Any other rational section is of the form $f \cdot S$ with $f \in \Bbbk(X)$, so $(fs) = (s) + (f)$, so the image of (s) in $\operatorname{Pic} X$ is independent of s. This gives the correspondence from line bundles to $\operatorname{Pic} X$.

Example. $\mathcal{L}$ =tautolocial bundle on $\mathbb{P}^1$. So fiber over a point $(x_0 \colon x_1) \in \mathbb{P}^1$ is $\Bbbk \cdot (x_0 \colon x_1) \subset \Bbbk^2$. Consider the section $(1, \dfrac{x_1}{x_0})$ is regular away from ∞ and has a first-order pole at ∞, no other zeroes or poles. So its divisor class is $[-\infty] \in \operatorname{Pic} \mathbb{P}^1$.

28. December 11

28.1. **Line bundles continued.** Last time: X =smooth irreducible variety, and line bundles on X up to isomorphism$\cong \operatorname{Pic} X$. Explicitly, both correspond to $H^1(X, \mathcal{O}_X^x)$.

The more direct way of constructing this correspondence is the following. Given a line bundle L on X, take a rational function $s \colon (s) \in \operatorname{Div} X$. The image of (s) in $\operatorname{Pic} X$ is indepenent of the choice of s (depends only on L).

Example. $\mathbb{P}^1$ has a tautological line bundle L: over each $(x_0 \colon x_1)$ associate the line $\Bbbk \cdot (x_0 \colon x_1) \subset \Bbbk$. So we can choose a section by $(1 \colon \dfrac{x_1}{x_0})$ over each $(x_0 \colon x_1)$. This rational sections has a single pole, so its divisor is $-[\text{point}] \in \operatorname{Pic} \mathbb{P}^1$ for some point in $\mathbb{P}^1$.

For $\mathbb{P}^n$, the tautological line bunlde corresponds to $-[H] \in \operatorname{Pic} \mathbb{P}^n$ where $H \subset \mathbb{P}^n$ is a hyperplane.

Example. Take $\mathbb{P}^1$ and consider $(L^*)^{\otimes d}$.

A fiber over $(x_0 \colon x_1)$ is a homogeneous degree d forms on $\Bbbk \cdot (x_0 \colon x_1)$.

Sections of this line bundle are degree d homogeneous functions in x_0, x_1. Take any degree d homogeneous polynomial in x_0, x_1. It is a section of this line bundle, with no poles, but it will have d zeroes (counted with multiplicity).

The divisor class is $d \cdot [\text{pt}] \in \operatorname{Pic} \mathbb{P}^1$.

Remark. Group operations on line bundles: $\otimes$ is the group law, and $L \mapsto L^*$ is the inversion.

Example. Consider the tangent bunlde $T\mathbb{P}^1$. This is a line bundle.

What is the number of zeroes and poles of a vector field on $\mathbb{P}^1$?

Well, if z is a coordinate on $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$, then $\frac{d}{dz}$ has no zeroes or poles on $\mathbb{A}^1$. What happens at ∞? Let $\zeta = \frac{1}{z}$. Then $\frac{dz}{dz}\frac{d}{dz} = \frac{d}{d\zeta}$. Then $-\frac{1}{\zeta^2}\frac{d}{dz} = \frac{d}{d\zeta}$.

Hence, $\frac{d}{dz} = -\zeta^2 \frac{d}{dz}$. So the image of $T\mathbb{P}^1$ in $\operatorname{Pic} X$ is 2[point].

For the cotangent bundle, $T^*\mathbb{P}^1$, we get -2[point].

Remark (Notation)**.** $K =$"canonical divisor"="divisor of the canonical line bundle" (more precisely, "canonical class") $\Lambda^{\dim X}(T^*X)$.

28.2. **Linear systems.**

Definition. For a divisor D on X, $L(D) = \{f \in \Bbbk(X) \colon (f) + D \geq 0$, i.e. $(f) \geq -D\}$. $L(D)$ is a vector space, and we are interested in its dimension $l(d) = \dim L(D)$.

Example. Let $X = \mathbb{A}^1$. Then $D = \sum n_i x_i$. E.g. if $D = -\text{pt}$, then $(f) \geq$pt ammounts to saying $L(D) = \{$regular functions vanishing at that point$\}$. Hence, $l(D) = \infty$ since in this case $L(D) = (x - \text{pt})\cdot\Bbbk[X]$. In fact $l(D)$ is infinite, for any affine X with $\dim X > 0$.

Example. $X = \mathbb{P}^1$, let $D = k\cdot\text{pt}$, $k \in \mathbb{Z}$ (think of pt$= \infty$).

Think of $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$ with $z =$coordinate on $\mathbb{A}^1$. Then $L(D) = \{p \in \Bbbk[z] \colon \deg(p) \leq k\}$. So $l(D) = 1 + k$ for $k \geq 0$ and 0 otherwise.

Example. Find the dimension of $\{$regular vector fields on $\mathbb{P}^1\}$. Claim: that space is isomorphic to $L(D)$ where D is the divisor of a (rational) vector field τ on $\mathbb{P}^1$.

Proof. Consider τ, $D = (\tau)$. Other rational vector fields are $\Bbbk(X) \cdot \tau$. Then $f \cdot \tau$ is regular if and only if $0 \leq (f \cdot \tau) = (f) + D$.

So the dimension is 3 with basis $\frac{d}{dz}$, $z\frac{d}{dz}$ and $z^2\frac{d}{dz}$. □

Proposition. $L(D) \cong$Space of global sections of the corresponding line bundle ($\mathcal{L}_D$).

Remark. IF D_1 D_2, then $L(D_1) \cong L(D_2)$. If $D_1 = (g) + D_2$, then $f \mapsto g \cdot f$.

28.3. **The Riemann-Roch Theorem.** The Riemann-Roch Theorem gives a "kind of" an answer to the question $l(D) =$? if X is a projective curve.

From now on, assume that $X =$smooth projective curve (also irreducible).

Definition. The degree of a divisor $\sum n_i x_i$ is $\sum n_i$.

Proposition. For any $f \in \Bbbk(X)^\times$, $\deg(f) = 0$. This is the same as saying that the number of preimages of 0 with multiplicity equals the number of preimages of ∞ with multiplicity.

Corollary. $\deg\colon \operatorname{Div} X \to \mathbb{Z}$ *factors through* $\operatorname{Pic} X$.

Theorem (Riemann-Roch)**.** *For any $D \in \operatorname{Div} X$:*

$$l(D) - l(K - D) = l - g + \deg(D)$$

for a constant $g \geq 0$ depending on X. (Only $[D] \in \operatorname{Pic} X$ matters)

Remark. g is the *(algebraic) genus* of X.

Example. $D = 0$. Then $l(0) - l(K) = 1 - g + \deg(0)$. Then $l(0) = 1$ since there are only constant functions a projective variety. Then $l(K) = \dim\{$global regular 1-forms on $X\}$. Then $\deg(0) = 0$ and so $g = \dim\{$global regular 1-forms on $X\}$.

Over $\mathbb{C}$, $g =$topological genus.

Example. $D = K$. Then $l(K) - l(0) = 1 - g + \deg K$, so $\deg K = 2g - 2$. On $\mathbb{P}^1$, $g = 0$.

29. December 13

Last lecture: Riemann-Roch Theorem and Beyond!

Last time: X =smooth projective irreducible. Consider $D \in \text{Div}X$, and $L(D) = \{f \in \Bbbk(X) : (f) + D \geq 0\}$ =global sections of the corresponding line bundles. $l(D) = \dim L(D)$.

Example. $X = \mathbb{P}^n$, and let $D = d \cdot H$ where H =hyperplane at infinity= $\mathbb{P}^{n-1} = \mathbb{P}^n - \mathbb{A}^n$.

Then $L(D) = \{$polynomials in n variables of degree at most $d\}$.

On the other hand, this is the same as sections of $(L^*)^{\otimes d}$ where L is the tautological line bundle; these sections are degree d homogeneous polynomials in $n+1$ variables.

Classically: replace f by $(f) + D = \tilde{D}$: f is determined by its divisor up to a nowhere vanishing regular functio, which on $\mathbb{P}^n$ is up to multiplicative constant.

Now $\tilde{D}$ must be ≥ 0 (effective), and it must be rationally equivalent to D ($\tilde{D} - D = (f)$). In the example, we consider the space of degree d hypersurfaces(=effective divisors) in $\mathbb{P}^n$ where we think of $\mathbb{P}^n$ as projecitve space of lines in $L(D)$.

So classically, the problem for degree 2, for example, is considering the family of all quadrics in $\mathbb{P}^2$, which is called a *linear system* in $\mathbb{P}^2$.

29.1. **Riemann-Roch Theorem.** Suppose X =curve. Then:

Theorem.

$$l(D) - l(K - D) = \deg D + 1 - g$$

where $0 \leq g = g(x)$ =*genus of* X*.*

Using the formula:

(1) $g = l(K) = \dim(\text{global 1-forms on } X)$
(2) $\deg K = 2g - 2$
(3) Riemann obtained the inequality $l(D) \geq \deg D + 1 - g$.
(4) $\deg(f) = 0$, so $\deg(D+(f)) = \deg D$. If $\deg D < 0$, then $D+(f) \ngeq 0$ (if $f \neq 0$). Hence, if $\deg D < 0$, then $l(D) = 0$.
(5) If $\deg D > 2g - 2 = \deg K$, then $\deg(K - D) < 0$, so $l(D) = \deg D + 1 - g$.

These give a nice graph of $l(D)$ vs. $\deg D$, which is a horizontal line at 0 until $\deg D = -1$, and a slope 1-line out of $(2g-1, g)$, and the inequality being things above the line and inside the square $(0,0)$ to $(0,g)$ to $(2g-1, g)$ to $(2g-1, 0)$.

Proposition. $l(D) < \infty$ for all D.

Proof. If $\deg D < 0$, then $l(D) = 0$. Let us show that $l(D) \leq l(D+x) \leq l(D) + 1$.

Note that $L(D) \subset L(D+x)$. SO we need to show that $\dim(L(D+x)/L(X)) \leq 1$. Suppose $D = n \cdot x +$other points. Then $f \in L(D+x)$ can have pole of order $n+1$ at x, as opposed to $f \in L(D)$ which can have pole of order n at x at most n. If t is a local coordinate at x, then $f = a_{n-1} \cdot t^{-n-1}$+lower order terms. Then f belongs to $L(D)$ if and only if $a_{-n-1} = 0$. So a_{-n-1} is a functional on $L(D+x)$, and $L(D)$ is the kernel. ☐

Note that this argument can be extended to any projectuve X and any divisor.

Now our diagram is restricted to the $l(D)$ vs. $\deg D$ relationship to a parallelogram.

Proof revisited. Set $\mathcal{O}(D) = \mathcal{O}_X(D)$ to be the sheaf of functions f such that $(f) + D \geq 0$. This is the same as the sheaf of sections of the corresponding line bundle. Then $\mathcal{O}(D) \subset \mathcal{O}(D+x)$ (subsheaf). So we have a short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{O}(D+x) \longrightarrow \mathcal{O}(D+x)/\mathcal{O}(D) \longrightarrow 0$$

Then $F = \mathcal{O}(D+x)/\mathcal{O}(D)$ is a sky-scraper sheaf: its stalk at x is 1-dimensional, and all other stalks are trivial.

Then we have a long exact sequence of cohomology:

$$0 \longrightarrow \Gamma(X, \mathcal{O}(D)) \longrightarrow \Gamma(X, \mathcal{O}(D+x)) \longrightarrow \Gamma(X, F)$$

The first two non-trivial parts are $L(D)$ and $L(D+x)$. ☐

29.2. **"Proof" of Riemann-Roch Theorem.**

Theorem. $H^i(X, \mathcal{O}(D)) = \begin{cases} L(D) = \Gamma(X, \mathcal{O}(D)) & i = 0 \\ L(K-D)^* & i = 1 \\ 0 & i > 1 \end{cases}$

So $l(D) - l(K-D) = \chi(\mathcal{O}_D)$ =Euler characteristic $\sum_i (-1)^i \dim H^i(X, \mathcal{O}(D))$.

Hence, $\chi(\mathcal{O}(D+x)) = \chi(\mathcal{O}(D)) + \chi(F)$. But $\chi(F) = 1$ because $H^i(X, F) = 0$. So in fact it equals $\chi(\mathcal{O}(D)) + 1$.

Hence ,$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \deg D$, and this is the Riemann-Roch theorem.

29.3. **Beyond curves.** Assume X =projective of dimension d. Let E be a vector bundle on X. We want to look at $H^i(X, E)$ (look at the sheaf of sections of E, locally free sheaf of X). The following are counterparts to the revisited proof.

Theorem (Grothendieck-Serre vanishing)**.** $\dim H^i(X, E) = 0$ *for* $i > \dim X$.

Theorem (Serre's finiteness theorem)**.** $\dim H^i(X, E) < \infty$.

Theorem (Serre's Duality)**.** $H^i(X, E) = H^{d-i}(X, E^* \otimes \Lambda^d T^* X)^*$

Theorem (Riemann-Roch-. . .)**.** $\chi(E) = \dim \sum (-1)^i \dim H^i(X, E)$ =*degree d polynoimal in E. In fact, threre is a closed formula...*

29.4. **Applications of Riemann-Roch.** Suppose X =projective curve

Example. Suppose $g = 0$. Then $l(D) = \deg D + 1 - g$ for $\deg D > 2g - 2 = -2$.

So $L(0)$ is 1-dimensional, so is just $\Bbbk$.

$L(\text{point})$ gives $\dim 2$, so there is a new function $f \in L(\text{point})$. This function is a regular map $f \colon Xt\o\mathbb{P}^1$, and it is easy to see that f is an isomorphism. So the only curve of genus 0 is the projective line.

Example. Conside $g = 1$. Then $l(D)\deg(D)$ provided $\deg(D) > 0$.

Then $L(0) = \Bbbk$ is of dimension 1.

Then $L(x)$ has dimension 1, so $L(x) = L(0) = \Bbbk$.

On the other hand, $L(2x)$ has dimension 2, so there is a new function $f \colon X \to \mathbb{P}^1$ which will be a double cover of $\mathbb{P}^1$ (this is the Weierstrass function).

For $L(3x)$ has dimension 3, so we have a new function $g = f'$.

Then for $L(4x) \ni f^2$, and $L(5x) \ni fg$, while $L(6x) \ni f^3, g^2$, but dimension is 7, so f and g satisfy a cubic relation.

Now, $X \overset{(f,g)}{\underset{\text{rational}}{\longrightarrow}} Y \subset \mathbb{A}^2$ where Y is a cubic. In fact, we have $X \to \bar{Y} \subset \mathbb{P}^2$, so any X is a (smooth) plane cubic. If $\bar{Y}$ was singular, X would be its resolution of singularities, but that would have genus 0.