SOME VT PROBLEMS (09/11/13)

1. Find, and give a proof of your answer, all positive integers n such that neither n nor n^2 contain a 1 when written in base 3.

2. Recall that the Fibonacci numbers $F(n)$ are defined by $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Determine the last digit of $F(2013)$ (e.g. the last digit of 2013 is 3).

(As you can guess, I updated this problem with a more relevant year.)

3. For each positive integer n, let S_n denote the total number of squares in an $n \times n$ square grid. Thus $S_1 = 1$ and $S_2 = 5$, because a 2×2 square grid has four 1×1 squares and one 2×2 square. Find a recurrence relation for S_n, and use it to calculate the total number of squares on a chess board (i.e. determine S_8).

4. Let n be a positive integer and let A be an $n \times n$ matrix with real numbers as entries. Suppose $4A^4 + I = 0$, where I denotes the identity matrix. Prove that the trace of A (i.e. the sum of the entries on the main diagonal) is an integer.

5. A set S of distinct positive integers has property **ND** if no element x of S divides the sum of the integers in any subset of $S \setminus \{x\}$. Here $S \setminus \{x\}$ means the set that remains after x is removed from S.

(i) Find the smallest positive integer n such that $\{3, 4, n\}$ has property **ND**.

(ii) If n is the number found in (i), prove that no set S with property **ND** has $\{3, 4, n\}$ as a proper subset.

(And now the real exercise: there are several vague and misleading sentences in this problem. See if you can spot them all.)

6. Find

$$\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \frac{x}{1 \times 2} + \frac{x^2}{2 \times 3} + \frac{x^3}{3 \times 4} + \dots$$

for $|x| < 1$.

7. Two diametrically opposite points P, Q lie on an infinitely long cylinder which has radius $2/\pi$. A piece of string with length 8 has its ends joined to P, is wrapped once round the outside of the cylinder, and then has its midpoint joined to Q (so there is length 4 of the string on each side of the cylinder). A paint brush is attached to the string so that it can slide along the full length the string. Find the area of the outside surface of the cylinder which can be painted by the brush.

UW Putnam Club

Meeting time: Wednesday 5–6:30pm, VV B 139.

Putnam competition: First Saturday in December (December 7, 2013). Two three-hour sessions of six problems each. Over 2,000 college students participate; there is also an official UW team (3 students).

Virginia Tech Regional Math Competition: 9–11:30 am, October 26, 2013, 7 problems. More than 600 contestants from over 100 schools. Kind of 'Putnam preparation', somewhat easier.

Typical topics: Linear algebra, elementary number theory, calculus, combinatorics; emphasis on problem-solving.