Basic methods of proof

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Argument by contradiction

1. **Euclid's theorem.** There are infinitely many prime numbers.

2. Prove that there is no polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

with integer coefficients and of degree at least 1 with the property that $P(0), P(1), P(2), \ldots$ are all prime numbers. (Open question: does there exist an integer coefficient polynomial $P(x)$ of degree at least 2 that takes infinitely many prime numbers as values? Also, google "Bunyakovsky conjecture".)

3. Determine all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ satisfying

$$xf(y) + yf(x) = (x + y)f(x + y).$$

Induction

1. **Fermat's little theorem.** Let p be a prime number, and n a positive integer. Then $n^p - n$ is divisible by p.

2. Prove that $3^n \geq n^3$ for all positive integers n.

3. Let n be a positive integer. Prove that

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{3}{2}.$$

4. Let $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be a strictly increasing function such that $f(2) = 2$ and $f(mn) = f(m)f(n)$ for every relatively prime pair of positive integers m and n. Prove that $f(n) = n$ for every positive integer n.

The pigeonhole principle

Pigeonhole principle. If $kn + 1$ objects are distributed among n boxes, one of the boxes will contain at least $k + 1$ objects.

1. Prove that every set of 10 two-digit integer numbers has two disjoint subsets with the same sum of elements.

2. Prove that every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a nonempty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n + 1}.$$

Exercise/Homework

1. Show that no set of nine consecutive integers can be partitioned into two sets with the product of the elements of the first set equal to the product of the elements of the second set.

2. Show that the interval $[0, 1]$ can not be partitioned into two disjoint sets A and B such that $B = A + a$ for some real number a.

3. Prove that in any group of six people there are either three mutual friends or three mutual strangers.

4. Let $n \geq 6$ be an integer. Show that
$$\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n.$$

5. Prove that the Fibonacci sequence ($F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$.) satisfies the identity
$$F_{2n+1} = F_{n+1}^2 + F_n^2, \quad \text{for } n \geq 0.$$

6. In how many ways can a $2 \times n$ rectangle be tiled with 2×1 dominoes? Give your answer in the form of a well-known function.

7. Prove that any positive integer can be represented as $\pm 1^2 \pm 2^2 \pm \cdots \pm n^2$ for some positive integer n and some choice of the signs.

8. A lattice point in the plane is a point (x, y) such that both x and y are integers. Find the smallest number n such that given n lattice points in the plane, there exist two whose midpoint is also a lattice point.

9. Let p be a prime number and a, b, c integers such that a and b are not divisible by p. Prove that the equation
$$ax^2 + by^2 = c \mod p$$
has integer solutions.