Putnam Club. Fall 2020

Problem session for October 7. Number theory-2.

Do not forget for problems 3,4,6 from the previous list.

1. Find all prime numbers p, such that $p^2 + 11$ has exactly 6 positive integer divisors.

2. Let $lcm(a, b)$ and $gcd(a, b)$ be the least common multiplier and the greatest common divisor of a and b. Prove that if $a \cdot gcd(a, b) + b \cdot lcm(a, b) < 5ab/2$, then $b|a$.

3. Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

4. Let $\mathbb{N}$ be the set of positive integers. Prove that there is no function $f : \mathbb{N} \to \mathbb{N}$, such that for all positive integers $n : 6f(f(n)) = 5f(n) - n$.

5. Let S be the smallest set of positive integers such that a) 2 is in S, b) n is in S whenever n^2 is in S, and c) $(n + 5)^2$ is in S whenever n is in S. Which positive integers are not in S? (The set S is "smallest" in the sense that S is contained in any other such set.)

6. Let k be positive integer, and $P(x)$, $Q(x)$ be polynomials with integer coefficients. Assume that for any integer x $P(Q(x)) - x$ is divisible by k. Prove that then $Q(P(x)) - x$ is also divisible by k for any x.

7. Is it possible to construct an infinite set M of positive integers in such a way that no element of M and no sum of several elements of M would be an exact power of an integer? (In other words M may not have elements of the form k^n, with integer k, $n > 2$, and no sums of elements of M may not be of this form either.)

8. Suppose p is a prime number and a sequence of integers is defined as follows: $a_0 = 0$, $a_1 = 1$ and $a_{k+2} = 2a_k - pa_{k-1}$ for $k \geq 0$. Find, with proof, all numbers p such that this consequence contains -1.

9. Prove that the sequence $2^n - 3$, $n \geq 1$, contains an infinite subsequence whose terms are pairwise relatively prime.