UW MADISON UNDERGRADUATE MATH COMPETITION

1. Fix $n \geq 3$. Start with a (planar convex) polygon with n sides, and draw all of its diagonals. What is the largest possible number of intersection points of the diagonals? (Count only inner intersection points, for example, if $n = 5$, there will be five intersections.)

2. $f(x)$ is a strictly positive, continuous function on the interval $2 \leq x \leq 4$. Compute

$$\int_2^4 \frac{f(x)}{xf(x) + xf(\frac{8}{x})} dx.$$

3. $P_1, \dots, P_{10}$ are ten points on the unit circle $x^2 + y^2 = 1$. What is the largest possible value of the quantity

$$\sum_{1 \leq i < j \leq 10} |P_i P_j|^2$$

(the sum of squares of pairwise distances between the points)?

4. Show that the equation

$$x^n + x = 1$$

has exactly one real solution in $[0, 1]$. Denote it by x_n. Show that

$$\lim_{n \to \infty} n(1 - x_n) = \infty.$$

5. (a) Written in binary, 2015 looks like

$$11111011111.$$

Find the smallest exponent $n > 0$ (if it exists) such that 2015^n ends in

$$\dots 1111111111$$

(ten ones) when written in binary.

(b) Written in binary, 2017 looks like

$$11111100001.$$

Find the smallest exponent $n > 0$ (if it exists) such that 2017^n ends in

$$\dots 00000000001$$

(ten zeros and one) when written in binary.

6. A is a square $n \times n$ matrix with integer entries. You are told that A is invertible; moreover, all entries of the inverse matrix A^{-1} are integers

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as well. Prove that the matrix $A + 10I$ has an inverse (however, it is not claimed that its entries are integers). Here I is the identity matrix.

7. A unit segment $[0, 1]$ is colored randomly using two colors, white and black, according to the following procedure. The segment starts white. On each step, we choose two random points a and b on the segment and switch the color of each point between them. (The points a and b are uniformly distributed on $[0, 1]$). We repeat this procedure n times, choosing two independent points on each step.

(a) What is the probability that the midpoint $1/2$ of the segment $[0, 1]$ is black after n steps? (n is arbitrary)

(b) What is the expected total length of the black region after n steps?