THIRD ANNUAL UW MADISON UNDERGRADUATE MATH COMPETITION

1. A beanstalk is continuously growing at a constant rate so that after one hour its length is doubled, after two hours it is four times the original length, after three hours it is eight times as long, and so one. (The growth is uniform: every part of the stalk grows at the same rate.) Two bugs start at a distance of one foot away from each other on the beanstalk and begin crawling towards each other at the constant speed x ft/hr (x is some number). For what values of x will the bugs meet?

2. Set

$$f(x) = \sum_{k=0}^{\infty} \frac{x+k}{e^k}.$$

Find

$$\int_0^2 f(x)e^{-x}\, dx.$$

3. There are ten real numbers $x_0, \ldots, x_9$ with $x_0 = 0$, $x_9 = 9$. What is the smallest possible value of the expression

$$(x_1 - x_0)^2 + \frac{(x_2 - x_1)^2}{2} + \frac{(x_3 - x_2)^2}{3} + \cdots + \frac{(x_9 - x_8)^2}{9} \quad ?$$

4. An *experiment* consists of tossing a coin ten times and recording the results (heads or tails), for instance, 'HTHHTHTTTT' could represent the result of an experiment. Two results are *similar* if they differ in no more than two place (or they are identical); thus, 'HTHHTHTTTT' is similar to 'TTHHTHTHTT', but not to 'TTTHTHTHTT'. Show that in a series of one hundred experiments, there will be two with similar results.

5. $p(x)$ is a polynomial with integer coefficients with the property that the equation $p(x) = a$ has an integral solution for $a = 1, 2, 3, \ldots, 10$. Show that the solutions must occur sequentially, either in ascending or descending order. That is, either there is an integer x such that $p(x) = 1$, $p(x+1) = 2, \ldots, p(x+9) = 10$, or there is an integer x such that $p(x) = 10$, $p(x+1) = 9, \ldots, p(x+9) = 1$.

6. Suppose $n > 0$ is fixed, and suppose S is a **finite** collection of $n \times n$ matrices with the following property: if $A, B \in S$, then $AB + A + B \in S$. Show that there exists a matrix $A \in S$ such that $A^2 + A = 0$.

7. 100 numbers are chosen uniformly from the interval $[0, 1]$, independently of each other.
(a) What is the probability that the second largest number is less than $1/2$?
(b) What is the expected value of the second largest number?

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