FOURTH ANNUAL UW MADISON UNDERGRADUATE MATH COMPETITION

1. Angela flips a fair coin 2018 times and Bert flips a fair coin 2017 times. What is the probability that Angela had more tails than Bert?

2. Consider a set S and a binary operation $\#$, i.e., for each $a, b \in S$, $a\#b \in S$. Assume $(a\#b)\#a = b$ for all $a, b \in S$. Prove that $a\#(b\#a) = b$ for all $a, b \in S$.

3. Let n be an odd positive integer. Show that the sum

$$1^n + 2^n + \cdots + n^n$$

is divisible by n^2.

4. Each of the six faces of a die is marked with an integer, not necessarily positive. The die is rolled 1000 times. Show that there is a time interval such that the product of all rolls in this interval is a cube of an integer. (For example, it could happen that the product of all outcomes between 5th and 20th throws is a cube; obviously, the interval has to include at least one throw!)

5. For which real numbers c does the inequality

$$e^{cx^2} \geq \frac{e^x + e^{-x}}{2}$$

hold for all real x?

6. Let b_n be the sequence of all positive integers such that the decimal expression for $\frac{1}{b_n}$ terminates in an *odd* digit:

$$1, 2, 4, 8, 10, \ldots$$

(For instance, 3 is not included because $\frac{1}{3} = 0.33\ldots$ does not terminate, 4 is included because $\frac{1}{4} = 0.25$ terminates in 5, which is odd; 5 is not included because $\frac{1}{5} = 0.2$ terminates in 2, which is even.)

Find

$$\sum \frac{1}{b_n}.$$

7. Compute

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \arctan\left(\frac{1+x^2}{1+y^2}\right) dx\, dy$$

Date: April 24, 2018.