Last updated on November 20, 2013.

(I2.1a) $A = -x^{-1} + C$,

(I2.1b) $B = -t^{-1} + C$,

(I2.1c) $C = tx^{-2} + C$,

(I2.1d) $I = \frac{1}{2}xt^2 + C$, $J = \frac{1}{2}x^2 t + C$.

(I4.3) $\int \sin^2 x \cos^2 x \, \mathrm{d}x = \frac{1}{4}\int \sin^2 2x \, \mathrm{d}x = \frac{1}{8}\int \bigl(1-\cos 4x\bigr) \, \mathrm{d}x = \frac{x}{8} - \frac{1}{32}\sin 4x + C$.

(I4.4) Rewrite the integral as

$$\int \cos^5\theta \, \mathrm{d}\theta = \int \cos^4\theta \, \underbrace{\cos\theta \, \mathrm{d}\theta}_{=\mathrm{d}\sin\theta}$$

and substitute $u = \sin\theta$. We get

$$\begin{aligned}\int \cos^5\theta \, \mathrm{d}\theta &= \int \bigl(1-u^2\bigr)^2 \, \mathrm{d}u \\ &= \int \bigl(1 - 2u^2 + u^4\bigr) \, \mathrm{d}u \\ &= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C \\ &= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C.\end{aligned}$$

For a different solution see the section on reduction formulas.

(I4.5) Hopefully you remembered that $\cos^2\theta + \sin^2\theta = 1$. The answer is $\theta + C$.

(I4.6) Use $2\sin A \sin B = \cos(A-B) - cos(A+B)$ to rewrite the integrand as $\sin x \sin 2x = \frac{1}{2}\bigl(\cos(x) - \cos(3x)\bigr)$. You then get

$$\int \sin x \sin 2x = \frac{1}{2}\int \bigl(\cos(x) - \cos(3x)\bigr) \, \mathrm{d}x = \frac{1}{2}\sin(x) - \frac{1}{6}\sin(x) + C.$$

(I7.1) $\displaystyle\int x^n \ln x \, \mathrm{d}x = \frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C.$

(I7.2) $\displaystyle\int e^{ax}\sin bx \, \mathrm{d}x = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C.$

(I7.3) $\displaystyle\int e^{ax}\cos bx \, \mathrm{d}x = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C.$

(I7.6) $\int_0^\pi \sin^{14} x \mathrm{d}x = \frac{13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1}{14\cdot 12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2}\frac{\pi}{2}$

(I7.7b) $\int_0^\pi \sin^5 x \mathrm{d}x = \frac{4\cdot 2}{5\cdot 3}\cdot 2$
$\int_0^\pi \sin^6 x \mathrm{d}x = \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\pi$
$\int_0^\pi \sin^7 x \mathrm{d}x = \frac{6\cdot 4\cdot 2}{7\cdot 5\cdot 3}\cdot 2$

(I7.8) $\int \cos^n x \mathrm{d}x = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \mathrm{d}x$;

$$\int \cos^4 x \, \mathrm{d}x = \frac{3}{8}x + \frac{1}{4}\cos^3 x \, \sin x + \frac{3}{8}\cos x \, \sin x + C$$

Setting $x = 0$ and $x = \pi/4$ leads to $\int_0^{\pi/4} \cos^4 x \mathrm{d}x = \frac{1}{4} + \frac{3}{32}\pi$

(I7.9) Hint: first integrate x^m.

(I7.10) $x \ln x - x + C$

(I7.11) $x(\ln x)^2 - 2x \ln x + 2x + C$

(I7.13) Substitute $u = \ln x$.

(I7.14) $\int_0^{\pi/4} \tan^5 x \mathrm{d}x = \frac{1}{4}(1)^4 - \frac{1}{2}(1)^2 + \int_0^{\pi/4} \tan x \mathrm{d}x = -\frac{1}{4} + \ln \frac{1}{2}\sqrt{2}$

(I7.17) Substitute $u = 1 + x^2$.

(I9.1a) $1 + \frac{4}{x^3-4}$

(I9.1b) $1 + \frac{2x+4}{x^3-4}$

(I9.1c) $1 - \frac{x^2+x+1}{x^3-4}$

(I9.1d) $\frac{x^3-1}{x^2-1} = x + \frac{x-1}{x^2-1}$. You can simplify this further: $\frac{x^3-1}{x^2-1} = x + \frac{x-1}{x^2-1} = x + \frac{1}{x+1}$.

(I9.2a) $x^2 + 6x + 8 = (x+3)^2 - 1 = (x+4)(x+2)$ so $\frac{1}{x^2+6x+8} = \frac{1/2}{x+2} + \frac{-1/2}{x+4}$ and

$$\int \frac{\mathrm{d}x}{x^2 + 6x + 8} = \frac{1}{2}\ln(x+2) - \frac{1}{2}\ln(x+4) + C.$$

(I9.2b) $\displaystyle\int \frac{\mathrm{d}x}{x^2 + 6x + 10} = \arctan(x+3) + C.$

(I9.2c) $\displaystyle\frac{1}{5}\int \frac{\mathrm{d}x}{x^2 + 4x + 5} = \frac{1}{5}\arctan(x+2) + C$

(I9.3) Multiply both sides of the equation

$$\frac{x^2+3}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

with $x(x+1)(x-1)$ and expand

$$\begin{aligned} x^2 + 3 &= A(x+1)(x-1) + Bx(x-1) + Cx(x+1) \\ &= A(x^2-1) + B(x^2-x) + C(x^2+x) \\ &= (A+B+C)x^2 + (C-B)x - A. \end{aligned}$$

Comparing coefficients of like powers of x tells us that

$$\begin{cases} A+B+C=1 \\ C-B=0 \\ -A=3 \end{cases}$$

Therefore $A=-3$ and $B=C=2$, i.e.

$$\frac{x^2+3}{x(x+1)(x-1)} = -\frac{3}{x} + \frac{2}{x+1} + \frac{2}{x-1}$$

and hence

$$\int \frac{x^2+3}{x(x+1)(x-1)}\,\mathrm{d}x = -3\ln|x| + 2\ln|x+1| + 2\ln|x-1| + \text{constant}.$$

(I9.4) To solve

$$\frac{x^2+3}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1},$$

multiply by x:

$$\frac{x^2+3}{(x+1)(x-1)} = A + \frac{Bx}{x+1} + \frac{Cx}{x-1}$$

and set $x=0$ (or rather, take the limit for $x \to 0$) to get $A=-3$; then multiply by $x+1$:

$$\frac{x^2+3}{x(x-1)} = \frac{A(x+1)}{x} + B + \frac{C(x+1)}{x-1}$$

and set $x=-1$ (or rather, take the limit for $x \to -1$) to get $B=2$; finally multiply by $x-1$:

$$\frac{x^2+3}{x(x+1)} = \frac{A(x-1)}{x} + \frac{B(x-1)}{x+1} + C,$$

and set $x=1$ (or take the limit for $x \to -1$) to get $C=2$.

(I9.5) Apply the method of equating coefficients to the form

$$\frac{x^2+3}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

In this problem, the Heaviside trick can still be used to find C and B; we get $B=-3$ and $C=4$. Then

$$\frac{A}{x} - \frac{3}{x^2} + \frac{4}{x-1} = \frac{Ax(x-1)+3(x-1)+4x^2}{x^2(x-1)}$$

so $A=-3$. Hence

$$\int \frac{x^2+3}{x^2(x-1)}\,\mathrm{d}x = -3\ln|x| + \frac{3}{x} + 4\ln|x-1| + \text{constant}.$$

(I9.6) Hint: look at the numerator.

(I9.7) 36.

(I9.11) $\frac{1}{2}(x^2 + \ln|x^2 - 1|) + C$

(I9.12) $5\ln|x-2| - 3\ln|x-1|$.

(I9.13) $x - 2\ln|x-1| + 5\ln|x-2| + C$

(I9.14) $\frac{1}{4}\ln|e^x - 1| - \frac{1}{4}\ln|e^x + 1| + \frac{1}{2}\arctan(e^x) + C$

(I9.15) Let $e^x = z$. Then $e^x dx = dz$. Now we can substitute:

$$\int \frac{e^x dx}{\sqrt{1+e^{2x}}} = \int \frac{dz}{\sqrt{1+z^2}}$$

Since we see a $\sqrt{1+z^2}$ the natural substitution is $z = \tan\theta$. This yields $dz = \sec^2\theta d\theta$, and we get

$$\begin{aligned}
&= \int \frac{\sec^2\theta d\theta}{\sqrt{1+\tan^2\theta}} \\
&= \int \frac{\sec^2\theta d\theta}{\sec\theta} \\
&= \int \sec\theta d\theta \\
&= \ln|\sec\theta + \tan\theta| + C
\end{aligned}$$

Now $\tan\theta = z$ and so $\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+z^2}$.

$$= \ln|\sqrt{1+z^2} + z| + C.$$

(I9.16) $\arctan(e^x + 1) + C$

(I9.17) $x - \ln(1+e^x) + C$

(I9.20) $-\ln|x| + \frac{1}{x} + \ln|x-1| + C$

(I13.1) $\arcsin x + C$

(I13.2) Use the substitution $x = 2\sin\theta$ and $dx = 2\cos\theta d\theta$ to obtain:

$$\begin{aligned}
\int \frac{dx}{\sqrt{4-x^2}} &= \int \frac{2\cos\theta d\theta}{\sqrt{4-4\sin^2\theta}} \\
&= \int \frac{2\cos\theta d\theta}{2\cos\theta} \\
&= \int 1 d\theta \\
&= \theta + C \\
&= \arcsin(\frac{x}{2}) + C
\end{aligned}$$

(I13.3) There are several ways to solve this problem. The following solution involves trigonometric substitution. We set $x = \tan\theta$ and so $dx = \sec^2\theta d\theta$ and we get

$$\begin{aligned}\int \sqrt{1+x^2}dx &= \int \sqrt{1+\tan^2\theta}\sec^2\theta d\theta \\ &= \int \sec^3\theta d\theta\end{aligned}$$

Now we need to integrate $\sec^3\theta$, which is a tricky problem in its own right!

$$\begin{aligned}&= \int \frac{1}{\cos^3\theta}d\theta \\ &= \int \frac{\cos\theta}{\cos^4\theta}d\theta \\ &= \int \frac{\cos\theta}{(1-\sin^2\theta)^2}d\theta\end{aligned}$$

Now let $u = \sin\theta$ so $du = \cos\theta d\theta$ and we get:

$$= \int \frac{du}{(1-u^2)^2}$$

Now we have a partial fractions problem.

$$\begin{aligned}&= \int \frac{du}{(1+u)^2(1-u)^2} \\ &= \int \frac{Au+B}{(1+u)^2} + \frac{Cu+D}{(1-u)^2}du\end{aligned}$$

We want to rewrite $\frac{1}{(1+u)^2(1-u)^2} = \frac{Au+B}{(1+u)^2} + \frac{Cu+D}{(1-u)^2}$. We need to use equating coefficients. After expanding and then equating the numerators, we obtain $1 = (A+C)u^3 + (B-2A+2C+D)u^2 + (A-2B+C+2D)u + (B+D)$. Equating coefficients yields $\begin{cases} A+C &= 0 \\ B-2A+2C+D &= 0 \\ A-2B+C+2D &= 0 \\ B+D &= 1 \end{cases}$

So $C = -A$ and $D = 1-B$. Subsituting these into the two middle equations we get: $\begin{cases} B-2A+2(-A)+(1-B) &= 0 \\ A-2B+(-A)+2(1-B) &= 0 \end{cases}$ The first equation simplifies to $-4A+1=0$ so $A = \frac{1}{4}$ and so $C = \frac{-1}{4}$. The second equation to $2-4B=0$ so $B = \frac{1}{2}$ so $D = \frac{1}{2}$. We now complete the integral computation:

$$\begin{aligned}
&= \int \frac{\frac{1}{4}u + \frac{1}{4}}{(1+u)^2} + \frac{\frac{1}{4}u + \frac{3}{4}}{(1-u)^2} du \\
&= \frac{1}{4} \int \frac{1}{(1+u)} + \frac{1}{(1+u)^2} + \frac{1}{(1-u)} + \frac{1}{(1-u)^2} du \\
&= \frac{1}{4} \left[\ln|1+u| - \frac{1}{(1+u)} - \ln|1-u| + \frac{1}{(1-u)} \right] + C \\
&= \frac{1}{4} \left[\ln\left|\frac{1+u}{1-u}\right| - \frac{2u}{1-u^2} \right] + C \\
&= \frac{1}{4} \ln\left|\frac{1+u}{1-u}\right| - \left(\frac{1}{2}\right) \frac{u}{1-u^2} + C
\end{aligned}$$

Let's substitute back in with $u = \sin\theta$ to get:

$$\begin{aligned}
&= \frac{1}{4} \ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| - \left(\frac{1}{2}\right) \frac{\sin\theta}{1-\sin^2\theta} + C \\
&= \frac{1}{4} \ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| - \left(\frac{1}{2}\right) \frac{\sin\theta}{\cos^2\theta} + C \\
&= \frac{1}{4} \ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| - \left(\frac{1}{2}\right) \tan\theta\sec\theta + C
\end{aligned}$$

Finally, since $x = \tan\theta$, we have $\sin\theta = \frac{x}{\sqrt{1+x^2}}$, and $\sec\theta = \sqrt{1+x^2}$ yielding:

$$= \frac{1}{4} \ln\left|\frac{1+\frac{x}{\sqrt{1+x^2}}}{1-\frac{x}{\sqrt{1+x^2}}}\right| - \left(\frac{1}{2}\right) x\sqrt{1+x^2} + C$$

We remark that, using rules of logarithms and some other algebraic manipulations, this can be transformed into the formula appearing in the book: $\frac{1}{2}(x\sqrt{1+x^2} + \ln|x+\sqrt{1+x^2}|) + C$

(I13.4) First we complete the square to get

$$\begin{aligned} 2x - x^2 &= -(x^2 - 2x) \\ &= -((x-1)^2 - 1) \\ &= 1 - (x-1)^2. \end{aligned}$$

So we now return to the integral:

$$\int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$$

Sub $x - 1 = \sin\theta$ and so $dx = \cos\theta d\theta$ and we get

$$\begin{aligned} &= \int \frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} \\ &= \int \frac{\cos\theta d\theta}{\cos\theta} \\ &= \int d\theta \\ &= \theta + C \\ &= \arcsin(x-1) + C. \end{aligned}$$

(I13.7) We substitute $x = 2\sin\theta$ so that $dx = 2\cos\theta$ and get:

$$\begin{aligned} \int_{-1}^{1} \frac{dx}{\sqrt{4-x^2}} &= \int_{x=-1}^{x=1} \frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}} \\ &= \int_{x=-1}^{x=1} \frac{2\cos\theta}{2\cos\theta} d\theta \\ &= \int_{x=-1}^{x=1} d\theta \\ &= [\theta]_{x=-1}^{x=1} \\ &= \Big[\arcsin(\frac{x}{2})\Big]_{-1}^{1} \\ &= \arcsin\left(\frac{1}{2}\right) - \arcsin\left(\frac{-1}{2}\right) \\ &= \pi/6 - (-\pi/6) = \pi/3. \end{aligned}$$

(I13.13) $\frac{1}{3}\arctan(x+1) + C$

(I13.14)

$$3x^2 + 6x + 15 = 3\big(x^2 + 2x + 5\big) = 3\Big((x+1)^2 + 4\Big) = 12\Big(\big(\frac{x+1}{2}\big)^2 + 1\Big)$$

Therefore substitute $u = \frac{x+1}{2}$.

(I15.1) We use integration by parts with $f = x$ and $g' = \sin x$. Then $f' = 1$ and $g = -\cos x$. So we get:

$$\begin{aligned}\int_0^a x \sin x &= \int_0^a fg' \\ &= [fg]_0^a - \int_0^a f'g \\ &= [-x\cos x]_0^a - \int_0^a (-\cos x) dx \\ &= -a\cos a - 0 + \int_0^a \cos x dx \\ &= -a\cos a + [\sin x]_0^a \\ &= -a\cos a + \sin a - \sin 0 \\ &= -a\cos a + \sin a.\end{aligned}$$

(I15.2) $\int_0^a x^2 \cos x \mathrm{d}x = (a^2+2)\sin a + 2a\cos a$

(I15.3) $\int_3^4 \frac{x\,\mathrm{d}x}{\sqrt{x^2-1}} = \left[\sqrt{x^2-1}\right]_3^4 = \sqrt{15} - \sqrt{8}$

(I15.4) Use $u = 1 - x^2$ so that $du = -2xdx$ and so $\frac{-du}{2} = xdx$.

$$\begin{aligned}\int_{1/4}^{1/3} \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} &= \int_{x=1/4}^{x=1/3} \frac{1}{2}\frac{-du}{\sqrt{u}} \\ &= -\int_{x=1/4}^{x=1/3} \frac{1}{2} u^{-1/2} du \\ &= -\left[u^{1/2}\right]_{x=1/4}^{x=1/3} \\ &= -\left[(1-x^2)^{1/2}\right]_{1/4}^{1/3} \\ &= -\left[\left(1-(1/3)^2\right)^{1/2} - \left(1-(1/4)^2\right)^{1/2}\right] \\ &= -\left[\left(1-\frac{1}{9}\right)^{1/2} - \left(1-\frac{1}{16}\right)^{1/2} -\right] \\ &= \frac{\sqrt{15}}{4} - \frac{\sqrt{8}}{3}.\end{aligned}$$

(I15.5) same as previous problem after substituting $x = 1/t$

(I15.6) Complete the square and substitute $u = (x+1)/4$. See the next problem for details. $\frac{1}{2}\ln|x^2+2x+17| - \frac{1}{4}\arctan(\frac{x+1}{4}) + C$

(I15.6) Completing the square leads to

$$\int \frac{x\,\mathrm{d}x}{\sqrt{x^2+2x+17}} = \int \frac{x\,\mathrm{d}x}{\sqrt{(x+1)^2+16}} = \frac{1}{4}\int \frac{x\,\mathrm{d}x}{\sqrt{\left(\frac{x+1}{4}\right)^2+1}}\,.$$

This suggests the substitution $u = \frac{x+1}{4}$, i.e. $x = 4u-1$:

$$\int \frac{x\,\mathrm{d}x}{\sqrt{x^2+2x+17}} = \frac{1}{4}\int \frac{(4u-1)\cdot 4\mathrm{d}u}{\sqrt{u^2+1}} = 4\int \frac{u\,\mathrm{d}u}{\sqrt{u^2+1}} - \frac{\mathrm{d}u}{\sqrt{u^2+1}}.$$

The first integral is

$$\int \frac{u\,\mathrm{d}u}{\sqrt{u^2+1}} = \sqrt{u^2+1}(+C),$$

which one can find by substituting $v = u^2+1$. The second integral is in the list **??**. Thus we end up with

$$\begin{aligned}\int \frac{x\,\mathrm{d}x}{x^2+2x+17} &= 4\sqrt{\left(\frac{x+1}{4}\right)^2+1} - \ln\Big\{\frac{x+1}{4} + \sqrt{\left(\frac{x+1}{4}\right)^2+1}\Big\} + C\\ &= \sqrt{x^2+2x+17} - \ln\Big\{\frac{x+1}{4} + \sqrt{\left(\frac{x+1}{4}\right)^2+1}\Big\} + C.\end{aligned}$$

This can be further simplified to

$$\begin{aligned} &= \sqrt{x^2+2x+17} - \ln\Big\{\frac{x+1}{4} + \frac{1}{4}\sqrt{(x+1)^2+16}\Big\} + C\\ &= \sqrt{x^2+2x+17} - \ln\frac{x+1+\sqrt{(x+1)^2+16}}{4} + C\\ &= \sqrt{x^2+2x+17} - \ln\big\{x+1+\sqrt{x^2+2x+17}\big\} + \ln 4 + C\\ &= \sqrt{x^2+2x+17} - \ln\big\{x+1+\sqrt{x^2+2x+17}\big\} + \hat{C}.\end{aligned}$$

(I15.10) $\frac{1}{x} + \ln|x-1| - \ln|x+1| + C$

(I15.27) $x^2\ln(x+1) - \frac{1}{2}x^2 + x - \ln(x+1) + C$

(I15.30) Substitute $x = t^2$ and integrate by parts. Answer: $x^2\arctan(\sqrt{x}) - \sqrt{x} + \arctan(\sqrt{x}) + C$

(I15.33) $\tan(x) - \sec(x) + C$

(I15.35) $\frac{1}{4}\ln(\frac{(x+1)^2}{x^2+1}) + \frac{1}{2}\arctan(x) + C$

(II4.1)

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}x}{(2+x)^2} &= \lim_{M\to\infty}\int_0^M \frac{\mathrm{d}x}{(2+x)^2}\\ &= \lim_{M\to\infty}\left[\frac{-1}{2+x}\right]_{x=0}^M\\ &= \lim_{M|to\infty}\left[\frac{-1}{2+M}-\frac{-1}{2+0}\right]\\ &= \frac{1}{2}.\end{aligned}$$

(II4.2) The integrand becomes infinite as $x \to \frac{1}{2}$ so this is indeed an improper integral.

$$\begin{aligned}\int_0^{1/2}(2x-1)^{-3}\,\mathrm{d}x = \lim_{a\nearrow\frac{1}{2}}\int_0^a (2x-1)^{-3}\,\mathrm{d}x &= \lim_{a\nearrow\frac{1}{2}}\left[-\frac{1}{4}(2x-1)^{-2}\right]_0^a\\ &= \lim_{a\nearrow\frac{1}{2}} -\frac{1}{4}(2a-1)^{-2}+\frac{1}{4} = -\infty.\end{aligned}$$

(II4.3) The integral is improper at $x=3$:

$$\begin{aligned}&\int_0^3 \frac{\mathrm{d}x}{\sqrt{3-x}}\\ &\quad= \lim_{a\nearrow 3}\int_0^a \frac{\mathrm{d}x}{\sqrt{3-x}}\\ &\quad= \lim_{a\nearrow 3}\left[-2\sqrt{3-x}\right]_{x=0}^{x=a}\\ &\quad= \lim_{a\nearrow 3}\left(-2\sqrt{3-a}+2\sqrt{3-0}\right)\\ &\quad= 2\sqrt{3}.\end{aligned}$$

(II4.4) 2

(II4.5) Substitute $u = x^2$ to get the antiderivative $\int xe^{-x^2}\,\mathrm{d}x = -\frac{1}{2}e^{-x^2}+C$. The improper integral then is:

$$\int_0^\infty xe^{-x^2}\,\mathrm{d}x = \lim_{M\to\infty}\int_0^M xe^{-x^2}\,\mathrm{d}x = \lim_{M\to\infty}\left(-\frac{1}{2}e^{-M^2}+\frac{1}{2}e^{-0^2}\right) = \frac{1}{2}.$$

(II4.8) $\int_0^7 x^{-1/2}\mathrm{d}x = \lim_{a\searrow 0}\int_a^7 x^{-1/2}\mathrm{d}x = \lim_{a\searrow 0}\left[2x^{1/2}\right]_{x=a}^7 = 2\sqrt{7}.$

(II4.9) This integral is infinite.

(II4.12) $\frac{1}{2}e^{-10}$

(II4.13) 1

(II4.14) $\frac{1}{2}$.

(II4.15) To do the integral, substitute $x = u^2$. The answer is 2.

(II4.21) $\int_1^\infty \pi r^2 \,\mathrm{d}x = \pi \int_1^\infty x^{-2}\mathrm{d}x = \pi$

(II6.3) $\dfrac{x}{x^2+2x} = \dfrac{1}{x+2} < \dfrac{1}{x}$ for all $x > 0$, so "False."

(II6.4) True

(II6.5) False.

(II6.7) True because $\frac{x}{x^2+1} = \frac{1}{x+1}$. For $x > 0$ it is always true that $x + 1 > x$ and therefore $\frac{1}{x} < \frac{1}{x+1}$ is true for all $x > 0$.

(II6.8) False.

(II6.9) True.

(II6.10) False.

(II6.11) True.

(II6.12) The integrand is $f(u) = \frac{u^2}{(u^2+1)^2}$, which is continuous at all $u \geq 0$. Therefore the integral is improper at $u \to \infty$, but nowhere else.

To determine if the integral converges we may therefore look at "the tail," i.e. we can look at

$$I = \int_1^\infty \frac{u^2}{\left(u^2+1\right)^2}\mathrm{d}u$$

instead of the integral starting at $u = 0$.

For all u we have $1 + u^2 \geq u^2$, and therefore

$$\frac{u^2}{\left(1+u^2\right)^2} \leq \frac{u^2}{(u^2)^2} = \frac{1}{u^2}.$$

Therefore

$$(\dagger) \qquad \int_1^\infty \frac{u^2}{\left(1+u^2\right)^2}\mathrm{d}u \leq \int_1^\infty \frac{\mathrm{d}u}{u^2} = 1.$$

This implies for the integral in the problem that

$$\int_0^\infty \frac{u^2}{\left(1+u^2\right)^2}\mathrm{d}u < \infty.$$

In other words the integral converges. To get an actual estimate for how big the integral is we already have (†), so we need to estimate the integral for $0 < u < 1$. On this interval we have

$$\frac{u^2}{\left(1+u^2\right)^2} \leq \frac{u^2}{(1)^2} = u^2,$$

so that

$$(\ddagger)\qquad \int_0^1 \frac{u^2}{\left(1+u^2\right)^2}\mathrm{d}u \le \int_0^1 u^2\,\mathrm{d}u = \frac{1}{3}.$$

Therefore

$$\int_0^\infty \frac{u^2}{\left(1+u^2\right)^2}\mathrm{d}u = \int_0^1 \frac{u^2}{\left(1+u^2\right)^2}\mathrm{d}u + \int_1^\infty \frac{u^2}{\left(1+u^2\right)^2}\mathrm{d}u \le \frac{1}{3} + 1 = \frac{4}{3}.$$

(II6.13) For large values of u the integrand is approximately given by

$$\frac{u^3}{\left(u^2+1\right)^2} \approx \frac{u^3}{(u^2)^2} = \frac{1}{u}.$$

Since the integral $\int_1^\infty \frac{\mathrm{d}u}{u}$ diverges, we expect the integral I to diverge too. To confirm this we try to "estimate the tail from below."

For $u > 1$ we have

$$1 + u^2 < 2u^2,$$

which implies

$$\frac{u^3}{\left(u^2+1\right)^2} > \frac{u^3}{(2u^2)^2} = \frac{1}{4u}.$$

Therefore

$$\int_1^\infty \frac{u^3}{\left(u^2+1\right)^2}\mathrm{d}u \ge \int_1^\infty \frac{\mathrm{d}u}{4u} = \infty.$$

It follow that

$$\int_0^\infty \frac{u^3}{\left(u^2+1\right)^2}\mathrm{d}u = \int_0^1 \frac{u^3}{\left(u^2+1\right)^2}\mathrm{d}u + \int_1^\infty \frac{u^3}{\left(u^2+1\right)^2}\mathrm{d}u \ge \int_1^\infty \frac{\mathrm{d}u}{4u} = \infty.$$

In other words the integral in the problem diverges.

(III4.1) General solution $y(x) = Ce^{\frac{1}{2}x^2}$; initial conditions imply $C = -e^{-2}$.

(III4.2) Implicit form of the solution $\tan y = -\frac{x^2}{2} + C$, so $C = \tan \pi/3 = \sqrt{3}$.

Solution $y(x) = \arctan\left(\sqrt{3} - x^2/3\right)$

(III4.3) Implicit form of the solution: $y + \frac{1}{2}y^2 + x + \frac{1}{2}x^2 = A + \frac{1}{2}A^2$. If we solve for y we get

$$y = -1 \pm \sqrt{A^2 + 2A + 1 - x^2 - 2x}$$

Whether we need the "+" or "−" depends on A.

(III4.4) Implicit form of the solution $\frac{1}{3}y^3 + \frac{1}{4}x^4 = C$; $C = \frac{1}{3}A^3$. Solution is $y = \sqrt[3]{A^3 - \frac{3}{4}x^4}$.

(III4.5) Integration gives $\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = x+C$. Solve for y to get $\frac{y-1}{y+1} = \pm e^{2x+2C} = \left(\pm e^{2C}\right)e^{2x}$.

Let $B = \pm e^{2C}$ be the new constant and we get $\dfrac{y-1}{y+1} = Be^{2x}$ whence $y = \dfrac{1+Be^{2x}}{1-Be^{2x}}$.

The initial value $y(0) = A$ tells us that $B = \frac{A-1}{A+1}$, and therefore the solution with initial value $y(0) = A$ is $y = \dfrac{A+1+(A-1)e^{2x}}{A+1-(A-1)e^{2x}}$.

(III4.6) $y(x) = \tan\bigl(\arctan(A) - x\bigr)$.

(III4.7) $y = \sqrt{2(x - \frac{x^3}{3}) + 1}$

(III6.5) $y = Ce^{-2x} - \frac{1}{3}e^x$

(III6.6) $y = xe^{\sin x} + Ae^{\sin x}$

(III6.8) $m(x) = \cos x$, $y(x) = \tan x + \dfrac{C}{\cos x}$, $C = 0$.

(III6.9) $m(x) = \dfrac{1}{\cos x}$, $y(x) = \frac{1}{2}\cos x \ln\dfrac{1+\sin x}{1-\sin x} + C\cos x$, $C = 0$.

(III6.10) Rewrite as $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{y}{\cos^2 x} = \dfrac{N}{\cos^2 x}$.

$m(x) = e^{\tan x}$, $y(x) = N + Ce^{-\tan x}$, $C = -N$.

(III6.11) Rewrite as $\dfrac{\mathrm{d}y}{\mathrm{d}x} - \dfrac{y}{x} = 1$.

$m(x) = \frac{1}{x}$, $y(x) = x\ln x + Cx$, $C = -\ln 2$.

(III6.15) $y = Ce^{-x^3/3}, \quad C = 5e^{1/3}$

(III6.16) $y = Ce^{-x-x^3}, \quad C = e^2$

(III9.1) (a) $\ln|y| = \frac{-t^2}{2} + C$ or y = 0, $y = e^C * e^{-t^2/2}$ Since e^C can be any positive value, and $-e^C$ can be any negative value, we can lump both together with the $y = 0$ solution to obtain: $y = A * e^{-t^2/2}$

Plugging in our initial value of $y(1.0) = 10.0$, we get $(10.0) = A * (e^{-(1)^2/2})$, so $A = 10 * e^{1/2}$

Solving for $y(3.0)$, $y(3.0) = 10e^{1/2} * e^{-((3.0)^2/2)} = 10e^{-4} \approx .183$

(b) There is only 1 step between $y(1.0)$ and $y(3.0)$. $t_0 = 1.0$, $y_0 = 10$, $m_0 = -10$, so $y(3.0) = 10 + (-10) * (2.0) = -10$

(c) If $h = 1.0$, then it takes $n = 2$ steps to reach $y(3.0)$, so $t_0 = 1.0$, $y_0 = 10$, $m_0 = -10$ $y_1 = y(2.0) = 10 + (-10) * (1.0) = 0$

$y_1 = 0$, $t_1 = 2.0$, $m_1 = -(2.0) * 0 = 0$,

$y_2 = y(3.0) = 0 + (0) * (1.0) = 0$

In tabular form:

t_k	y_k	$m_k = f(t_k, y_k)$	$y_{k+1} = y_k + m_k h$
$t_0 = 1.0$	$y_0 = 10$	$-(1) * (10) = -10$	$10 + (-10)(1.0) = 0$
$t_1 = 2.0$	$y_1 = 0$	$-(2.0) * 0 = 0$	$0 + (0)(1.0) = 0$
$t_2 = 3.0$	$y_2 = 0$		

Now we can do the same thing with steps of $h = 2/3$ and $h = .5$

(h=2/3)

t_k	y_k	$m_k = f(t_k, y_k)$	$y_{k+1} = y_k + m_k h$
$t_0 = 1.0$	$y_0 = 10$	$-(1.0) * (10) = -10$	$10 + (-10)(2/3) = 10/3$
$t_1 = 5/3$	$y_1 = 10/3$	$-(5/3) * 10/3 = -50/9$	$10/3 + (-50/9)(2/3) = -10/27$
$t_2 = 7/3$	$y_2 = -10/27$	$-(7/3) * (-10/27) = 70/81$	$-10/27 + (70/81)(2/3) = 50/243$
$t_3 = 3.0$	$y_3 = 50/243$		

(h=1/2)

t_k	y_k	$m_k = f(t_k, y_k)$	$y_{k+1} = y_k + m_k h$
$t_0 = 1.0$	$y_0 = 10$	$-(1.0) * (10) = -10$	$10 + (-10)(1/2) = 5$
$t_1 = 3/2$	$y_1 = 5$	$-(3/2) * 5 = -15/2$	$5 + (-15/2)(1/2) = 5/4$
$t_2 = 2$	$y_2 = 5/4$	$-(2) * (5/4) = -5/8$	$5/4 + (-5/8)(1/2) = 15/16$
$t_3 = 5/2$	$y_3 = 15/16$	$-(5/2) * (15/16) = -75/32$	$15/16 + (-75/32)(1/2) = -15/64$
$t_4 = 3$	$y_4 = -15/64$		

(III9.2) (a) If $h = 1$, then there is only one step between $y(0)$ and $y(1)$, therefore: $x_0 = 0$, $y_0 = 1$, $m_0 = 1$, and $e = y_1 = 1 + (1) * (1) = 2$

(h=1/2) If $h = 1/2$, then x_0, y_0, and m_0 are the same, and $y_1 = 1+(1)*(1/2) = 3/2$ $x_1 = 1/2$, $y_1 = 3/2$, $m_1 = 3/2$, and $e \approx y_2 = 3/2 + (3/2) * (1/2) = 9/4$

(h=1/3) If $h = 1/3$, then x_0, y_0, and m_0 are the same, and $y_1 = 1+(1)*(1/3) = 4/3$ $x_1 = 1/3$, $y_1 = 4/3$, $m_1 = 4/3$, and $y_2 = 4/3 + (4/3) * (1/3) = 16/9$ $x_2 = 2/3$, $y_2 = 16/9$, $m_2 = 16/9$, and $e \approx y_3 = 16/9 + (16/9) * (1/3) = 64/27 \approx 2.37$

(h=1)

x_k	y_k	$m_k = f(x_k, y_k)$	$y_{k+1} = y_k + m_k h$
$x_0 = 0$	$y_0 = 1$	(1)	$1 + (1)(1) = 2$
$x_1 = 1$	$y_1 = 2$		

(h=1/2)

x_k	y_k	$m_k = f(x_k, y_k)$	$y_{k+1} = y_k + m_k h$
$x_0 = 0$	$y_0 = 1$	(1)	$1 + (1)(1/2) = 3/2$
$x_1 = 1/2$	$y_1 = 3/2$	3/2	$3/2 + (3/2)(1/2) = 9/4$
$x_2 = 1$	$y_2 = 9/4$		

(h=1/3)	x_k	y_k	$m_k = f(x_k, y_k)$	$y_{k+1} = y_k + m_k h$
	$x_0 = 0$	$y_0 = 1$	(1)	$1 + (1)(1/3) = 4/3$
	$x_1 = 1/3$	$y_1 = 4/3$	$4/3$	$4/3 + (4/3)(1/3) = 16/9$
	$x_2 = 2/3$	$y_2 = 16/9$	$16/9$	$16/9 + (16/9)(1/3) = 64/27$
	$x_3 = 1$	$y_3 = 64/27$		

b. Hint: $9/4 = 3^2/2^2$ and $64/27 = 4^3/3^3$

(III11.3a) Let $X(t)$ be the rabbit population size at time t. The rate at which this population grows is dX/dt rabbits per year.

$\frac{5}{100}X$ from growth at 5% per year
$\frac{2}{100}X$ from death at 2% per year
-1000 car accidents
$+700$ immigration from Sun Prairie

Together we get

$$\frac{dX}{dt} = \frac{3}{100}X - 300.$$

This equation is both separable and first order linear, so we can choose from two methods to find the general solution, which is

$$X(t) = 10,000 + Ce^{0.03t}.$$

If $X(1991) = 12000$ then

$$10,000 + Ce^{0.03\cdot 1991} = 12,000 \implies C = 2,000e^{-0.03\cdot 1991}\text{(don't simplify yet!)}$$

Hence

$$X(1994) = 10,000 + 2,000e^{-0.03\cdot 1991}e^{0.03\cdot 1994} = 10,000 + 2,000e^{0.03\cdot(1994-1991)} = 10,000 + 2,000e^{0.09} \approx 12,188.\ldots$$

(III11.4a) $\sec^{-1}$.

(III11.4b) $T_s(t) = T_r + Ce^{-Kt}$. C has the same units as T_r and T_s, so C is a temperature: degree Fahrenheit. $\lim_{t\to\infty} T_s(t) = T_r$: in the long run the soup will approach room temperature.

(III11.4c) (ii) Given $T_s(0) = 180$, $T_r = 75$, and $T_s(5) = 150$. This gives the following equations:

$$T_r + C = 180, \quad T_r + Ce^{-5K} = 105 \implies C = 105, \quad -5K = \ln\frac{30}{105} = \ln\frac{2}{7} = -\ln\frac{7}{2}.$$

When is $T_s = 90$? Solve $T_s(t) = 90$ for t using the values for T_r, C, K found above (K is a bit ugly so we substitute it at the end of the problem):

$$T_s(t) = T_r + Ce^{-Kt} = 75 + 105e^{-Kt} = 90 \implies e^{-Kt} = \frac{15}{105} = \frac{1}{7}.$$

Hence

$$t = \frac{\ln 1/7}{-K} = \frac{\ln 7}{K} = \frac{\ln 7}{\ln 7/2}.$$

(III11.6) (a) Let $y(t)$ be the amount of "retaw" (in gallons) in the tank at time t. Then

$$\frac{dy}{dt} = \underbrace{\frac{5}{100}y}_{\text{growth}} - \underbrace{3}_{\text{removal}}.$$

(b) $y(t) = 60 + Ce^{t/20} = 60 + (y_0 - 60)e^{t/20}$.

(c) If $y_0 = 100$ then $y(t) = 60 + 40e^{t/20}$ so that $\lim_{t\to\infty} y(t) = +\infty$.

(d) $y_0 = 60$.

(III11.7) Finding the equation is the hard part. Let $A(t)$ be the ***volume*** of acid in the vat at time t. Then $A(0) = 25\%$ of $1000 = 250$gallons.

$A'(t) =$ the volume of acid that gets pumped in minus the volume that gets extracted per minute. Per minute 40% of 20 gallons, i.e. 8 gallons of acid get added. The vat is well mixed, and $A(t)$ out of the 1000gallons are acid, so if 20 gallons get extracted, then $\frac{A}{1000} \cdot 20$ of those are acid. Hence

$$\frac{dA}{dt} = 8 - \frac{A}{1000} \cdot 20 = 8 - \frac{A}{50}.$$

The solution is $A(t) = 400 + Ce^{-t/50} = 400 + (A(0) - 400)e^{-t/50} = 400 - 150e^{-t/50}$.

The ***concentration*** at time t is

$$\text{concentration} = \frac{A(t)}{\text{total volume}} = \frac{400 - 150e^{-t/50}}{1000} = 0.4 - 0.15r^{-t/50}.$$

If we wait for very long the concentration becomes

$$\text{concentration} = \lim_{t\to\infty} \frac{A(t)}{1000} = 0.4.$$

(III11.8c) P is the volume of polluted water in the lake at time t. At any time the fraction of the lake water that is polluted is P/V, so if 24 cubic feet are drained then $\frac{P}{V} \cdot 24$ of those are polluted. Here $V = 10^9$; for simplicity we'll just write V until the end of the problem. We get

$$\frac{dP}{dt} = \text{"in minus out"} = 3 - \frac{P}{V} \cdot 24$$

whose solution is $P(t) = \frac{1}{8}V + Ke^{-\frac{24}{V}t}$. Here K is an arbitrary constant (which we can't call C because in this problem C is the concentration).

The concentration at time t is

$$C(t) = \frac{P(t)}{V} = \frac{1}{8} + \frac{K}{V}e^{-\frac{24}{V}t} = \frac{1}{8} + \big(C_0 - \frac{1}{8}\big)e^{-\frac{24}{V}t}.$$

No matter what C_0 is we always have

$$\lim_{t\to\infty} C(t) = 0$$

because $\lim_{t\to\infty} e^{-\frac{24}{V}t} = 0$.

If $C_0 = \frac{1}{8}$ then the concentration of polluted water remains constant: $C(t) = \frac{1}{8}$.

(IV4.1) Use Taylor's formula : $Q(x) = 43 + 19(x-7) + \frac{11}{2}(x-7)^2$.

A different, correct, but more laborious (clumsy) solution is to say that $Q(x) = Ax^2 + Bx + C,$, compute $Q'(x) = 2Ax + B$ and $Q''(x) = 2A$. Then

$$Q(7) = 49A + 7B + C = 43, \qquad Q'(7) = 14A + B = 19, \qquad Q''(7) = 2A = 11.$$

This implies $A = 11/2$, $B = 19 - 14A = 19 - 77 = -58$, and $C = 43 - 7B - 49A = 179\frac{1}{2}$.

(IV4.2) $p(x) = 3 + 8(x-2) - \frac{1}{2}(x-2)^2$

(IV4.17) $T_\infty e^t = 1 + t + \frac{1}{2!}t^2 + \cdots + \frac{1}{n!}t^n + \cdots$

(IV4.18) $T_\infty e^{\alpha t} = 1 + \alpha t + \frac{\alpha^2}{2!}t^2 + \cdots + \frac{\alpha^n}{n!}t^n + \cdots$

(IV4.19) $T_\infty \sin(3t) = 3t - \frac{3^3}{3!}t^3 + \frac{3^5}{5!}t^5 + \cdots + \frac{(-1)^k 3^{2k+1}}{(2k+1)!}t^{2k+1} + \cdots$

(IV4.20) $T_\infty \sinh t = t + \frac{1}{3!}t^3 + \cdots + \frac{1}{(2k+1)!}t^{2k+1} + \cdots$

(IV4.21) $T_\infty \cosh t = 1 + \frac{1}{2!}t^2 + \cdots + \frac{1}{(2k)!}t^{2k} + \cdots$

(IV4.22) $T_\infty \frac{1}{1+2t} = 1 - 2t + 2^2t^2 - \cdots + (-1)^n 2^n t^n + \cdots$

(IV4.23) $T_\infty \frac{3}{(2-t)^2} = \frac{3}{2^2} + \frac{3 \cdot 2}{2^3}t + \frac{3 \cdot 3}{2^4}t^2 + \frac{3 \cdot 4}{2^5}t^3 + \cdots + \frac{3 \cdot (n+1)}{2^{n+2}}t^n + \cdots$ (note the cancellation of factorials)

(IV4.24) $T_\infty \ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 + \cdots + \frac{(-1)^{n+1}}{n}t^n + \cdots$

(IV4.25) $T_\infty \ln(2+2t) = T_\infty \ln[2 \cdot (1+t)] = \ln 2 + \ln(1+t) = \ln 2 + t - \frac{1}{2}t^2 + \frac{1}{3}t^3 + \cdots + \frac{(-1)^{n+1}}{n}t^n + \cdots$

(IV4.26) $T_\infty \ln\sqrt{1+t} = T_\infty \frac{1}{2}\ln(1+t) = \frac{1}{2}t - \frac{1}{4}t^2 + \frac{1}{6}t^3 + \cdots + \frac{(-1)^{n+1}}{2n}t^n + \cdots$

(IV4.27) $T_\infty \ln(1+2t) = 2t - \frac{2^2}{2}t^2 + \frac{2^3}{3}t^3 + \cdots + \frac{(-1)^{n+1}2^n}{n}t^n + \cdots$

(IV4.28) $T_\infty \ln\sqrt{\left(\frac{1+t}{1-t}\right)} = T_\infty \left[\frac{1}{2}\ln(1+t) - \frac{1}{2}\ln(1-t)\right] = t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \cdots + \frac{1}{2k+1}t^{2k+1} + \cdots$

(IV4.29) $T_\infty \frac{1}{1-t^2} = T_\infty \left[\frac{1/2}{1-t} + \frac{1/2}{1+t}\right] = 1 + t^2 + t^4 + \cdots + t^{2k} + \cdots$ (we could also substitute $x = -t^2$ in the geometric series $1/(1+x) = 1 - x + x^2 + \cdots$, later in this chapter we will use "little-oh" to justify this point of view.)

(IV4.30) $T_\infty \frac{t}{1-t^2} = T_\infty \left[\frac{1/2}{1-t} - \frac{1/2}{1+t}\right] = t + t^3 + t^5 + \cdots + t^{2k+1} + \cdots$ (note that this function is t times the previous function so we would think its Taylor series is just t times the Taylor series of the previous function. Again, "little-oh" justifies this.)

(IV4.31) The pattern for the n^{th} derivative repeats every time we increase n by 4. So we indicate the the general terms for $n = 4m, 4m+1, 4m+2$ and $4m+3$:

$$T_\infty(\sin t + \cos t) = 1 + t - \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots + \frac{t^{4m}}{(4m)!} + \frac{t^{4m+1}}{(4m+1)!} - \frac{t^{4m+2}}{(4m+2)!} - \frac{t^{4m+3}}{(4m+3)!} + \cdots$$

(IV4.32) Use a double angle formula

$$T_\infty(2\sin t\cos t) = \sin 2t = 2t - \frac{2^3}{3!}t^3 + \cdots + \frac{2^{4m+1}}{(4m+1)!}t^{4m+1} - \frac{2^{4m+3}}{(4m+3)!}t^{4m+3} + \cdots$$

(IV4.33) $T_3 \tan t = t + \frac{1}{3}t^3$. There is no simple general formula for the n^{th} term in the Taylor series for $\tan x$.

(IV4.34) $T_\infty\left[1 + t^2 - \frac{2}{3}t^4\right] = 1 + t^2 - \frac{2}{3}t^4$

(IV4.35) $T_\infty[(1+t)^5] = 1 + 5t + 10t^2 + 10t^3 + 5t^4 + t^5$

(IV4.36) $T_\infty \sqrt[3]{1+t} = 1 + \frac{1/3}{1!}t + \frac{(1/3)(1/3-1)}{2!}t^2 + \cdots + \frac{(1/3)(1/3-1)(1/3-2)\cdots(1/3-n+1)}{n!}t^n + \cdots$

(IV4.37) $10! \cdot 2^6$

(IV4.38) Because of the addition formula

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

we should get the same answer for f and g, since they are the same function!

The solution is

$$T_\infty \sin(x+a) = \sin a + \cos(a)x - \frac{\sin a}{2!}x^2 - \frac{\cos a}{3!}x^3 + \cdots$$
$$\cdots + \frac{\sin a}{(4n)!}x^{4n} + \frac{\cos a}{(4n+1)!}x^{4n+1} - \frac{\sin a}{(4n+2)!}x^{4n+2} - \frac{\cos a}{(4n+3)!}x^{4n+3} + \cdots$$

(IV7.1)

$$f(x) = f^{(4)}(x) = \cos x, \qquad f^{(1)}(x) = f^{(5)}(x) = -\sin x,$$
$$f^{(2)}(x) = -\cos x, \qquad f^{(3)}(x) = \sin x,$$

so

$$f(0) = f^{(4)}(0) = 1, \qquad f^{(1)}(0) = f^{(3)}(0) = 0, \qquad f^{(2)}(0) = -1.$$

and hence the fourth degree Taylor polynomial is

$$T_4\{\cos x\} = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!}x^k = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$

The error is

$$R_4\{\cos x\} = \frac{f^{(5)}(\xi)}{5!}x^5 = \frac{(-\sin\xi)}{5!}x^5$$

for some ξ between 0 and x. As $|\sin\xi| \le 1$ we have

$$\left|\cos x - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)\right| = |R_4(x)| \le \frac{|x^5|}{5!} < \frac{1}{5!}$$

for $|x| < 1$.

Remark: Since the fourth and fifth order Taylor polynomial for the cosine are the same, it must be that $R_4(x) = R_5(x)$. It follows that $\frac{1}{6!}$ is also an upper bound.

(IV7.3a) The polynomial is $p(x) = 2 + \frac{1}{12}x - \frac{1}{9\cdot 32}x^2$. Then

$$p(1) \approx 2.07986111$$

and the error satisfies:

$$|\sqrt[3]{9} - p(1)| \le \frac{10}{27} \cdot 8^{-\frac{8}{3}} \cdot \frac{1}{3!} \approx 0.00024112654321$$

The $\sqrt[3]{9}$ according to a computer is:

$$\sqrt[3]{9} \approx 2.08008382305$$

(IV7.3b) The polynomial is $p(x) = 2 + \frac{1}{12}x - \frac{1}{9\cdot 32}x^2 + \frac{5}{27\cdot 256}x^3$. Then

$$p(1) \approx 2.087528935185185$$

and the error satisfies:

$$|\sqrt[3]{9} - p(1)| \le \frac{80}{81} \cdot 8^{-\frac{11}{3}} \cdot \frac{1}{4!} \approx 0.0000200938786008$$

The $\sqrt[3]{9}$ according to a computer is:

$$\sqrt[3]{9} \approx 2.087528935185185$$

(IV7.4) The polynomial is $p(x) = 3 + \frac{1}{6}x - \frac{1}{4\cdot 27}x^2$. Then

$$p(1) \approx 3.15740740740$$

and the error satisfies:

$$|\sqrt{10} - p(1)| \le \frac{3}{8} \cdot 9^{-\frac{5}{2}} \cdot \frac{1}{3!} \approx 0.0000005.2922149401$$

The $\sqrt{10}$ according to a computer is:

$$\sqrt{10} \approx 3.1622776601683$$

(IV9.2) Notice that

$$\frac{\left(1 + x^2\right)^2 - 1}{x^2} = 2 + x^2 \to 2$$

as $x \to 0$. Thus, $\left(1 + x^2\right)^2 - 1$ is not $o(x^2)$.

(IV9.3) Recall that

$$\sqrt{1+x} = 1 + \frac{1}{2}x + o(x)$$

and similarly that

$$\sqrt{1-x} = 1 + \frac{1}{2}(-x) + o(-x) = 1 - \frac{1}{2}x + o(x)$$

therefore

$$\frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \frac{x + o(x)}{x} \to 1$$

as $x \to 0$. Thus, $\sqrt{1+x} - \sqrt{1-x}$ is not $o(x)$.

(IV9.4) Suppose that $f(x)$ and $g(x)$ are both $o(x)$. This means that

$$\lim_{x\to 0} \frac{f(x)}{x} = 0 = \lim_{x\to 0} \frac{g(x)}{x}$$

Then we know that

$$\lim_{x\to 0} \frac{f(x)+g(x)}{x} = \lim_{x\to 0} \frac{f(x)}{x} + \lim_{x\to 0} \frac{g(x)}{x} = 0$$

Thus, $f(x)+g(x)$ is also $o(x)$. Since these are arbitrary $o(x)$ functions, we can say that $'o(x) + o(x) = o(x)'$.

(IV9.8) Notice that the functions x^3 and $-x^3$ are both $o(x^2)$, since

$$\lim_{x\to 0} \frac{x^3}{x^2} = 0 = \lim_{x\to 0} \frac{-x^3}{x^2}$$

But, $x^3 - (-x^3) = 2x^3$ is not $o(x^3)$, since

$$\lim_{x\to 0} \frac{2x^3}{x^3} = 2$$

(IV9.9) Suppose that $f(x) = o(2x)$. This means that

$$\lim_{x\to 0} \frac{f(x)}{2x} = 0$$

But then

$$2\lim_{x\to 0} \frac{f(x)}{2x} = \lim_{x\to 0} \frac{2f(x)}{2x} = \lim_{x\to 0} \frac{f(x)}{x} = 0$$

so that $f(x) = o(x)$.

(IV9.14a) We know from the Taylor expansion for $\sqrt{1+x}$ that

$$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 + o(x^2)$$

From this, we can see that $\sqrt{1+x^2} = 1 + o(x^k)$ exactly when $\frac{1}{2}x^2$ is $o(x^k)$, which is true when $k < 2$.

(IV9.14c) We know from the Taylor expansion of $\cos(x)$ that

$$\cos(x^2) = 1 - \frac{1}{2}x^4 + o(x^4)$$

so that

$$1 - \cos(x^2) = \frac{1}{2}x^4 + o(x^4)$$

which is $o(x^k)$ exactly when $k < 4$.

(IV9.15d) The PFD of g is $g(x) = \dfrac{1}{x-2} - \dfrac{1}{x-1}$.
$g(x) = \frac{1}{2} + \big(1 - \frac{1}{2^2}\big)x + \big(1 - \frac{1}{2^3}\big)x^2 + \cdots + \big(1 - \frac{1}{2^{n+1}}\big)x^n + \cdots$.
So $g_n = 1 - 1/2^{n+1}$ and $g^{(n)}(0)$ is $n!$ times that.

(IV9.16) You could repeat the computations from problem **??**, and this would get us the right answer with the same amount of work. In this case we could instead note that $h(x) = xg(x)$ so that

$$h(x) = \tfrac{1}{2}x + \big(1 - \tfrac{1}{2^2}\big)x^2 + \big(1 - \tfrac{1}{2^3}\big)x^3 + \cdots + \big(1 - \tfrac{1}{2^{n+1}}\big)x^{n+1} + \cdots$$

Therefore $h_n = 1 - 1/2^n$.

The PFD of $k(x)$ is

$$k(x) = \frac{2-x}{(x-2)(x-1)} \stackrel{\text{cancel!}}{=} \frac{1}{1-x},$$

the Taylor series of k is just the Geometric series.

(IV9.18) $T_\infty e^{at} = 1 + at + \dfrac{a^2}{2!}t^2 + \cdots + \dfrac{a^n}{n!}t^n + \cdots$.

(IV9.19) $e^{1+t} = e \cdot e^t$ so $T_\infty e^{1+t} = e + et + \frac{e}{2!}t^2 + \cdots + \frac{e}{n!}t^n + \cdots$

(IV9.20) Substitute $u = -t^2$ in the Taylor series for e^u.

$$T_\infty e^{-t^2} = 1 - t^2 + \frac{1}{2!}t^4 - \frac{1}{3!}t^6 + \cdots + \frac{(-1)^n}{n!}t^{2n} + \cdots$$

(IV9.21) PFD! The PFD of $\frac{1+t}{1-t}$ is $\frac{1+t}{1-t} = -1 + \frac{2}{1-t}$. Remembering the Geometric Series we get

$$T_\infty \frac{1+t}{1-t} = 1 + 2t + 2t^2 + 2t^3 + \cdots + 2t^n + \cdots$$

(IV9.22) Substitute $u = -2t$ in the Geometric Series $1/(1-u)$. You get

$$T_\infty \frac{1}{1+2t} = 1 - 2t + 2^2t^2 - 2^3t^3 + \cdots + \cdots + (-1)^n 2^n t^n + \cdots$$

(IV9.23) $f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n}$

(IV9.24)

$$T_\infty \frac{\ln(1+x)}{x} = \frac{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + (-1)^{n-1}\frac{1}{n}x^n + \cdots}{x}$$
$$= 1 - \frac{1}{2}x + \frac{1}{3}x^2 + \cdots + (-1)^{n-1}\frac{1}{n}x^{n-1} + \cdots$$

(IV9.25)

$$T_\infty \frac{e^t}{1-t} = 1+2t+\big(1+1+\tfrac{1}{2!}\big)t^2+\big(1+1+\tfrac{1}{2!}+\tfrac{1}{3!}\big)t^3+\cdots+\big(1+1+\tfrac{1}{2!}+\cdots+\tfrac{1}{n!}\big)t^n+\cdots$$

(IV9.26) $1/\sqrt{1-t} = (1-t)^{-1/2}$ so

$$T_\infty \frac{1}{\sqrt{1-t}} = 1 + \tfrac{1}{2}t + \frac{\frac{1}{2}\frac{3}{2}}{1\cdot 2}t^2 + \frac{\frac{1}{2}\frac{3}{2}\frac{5}{2}}{1\cdot 2\cdot 3}t^3 + \cdots$$

(be careful with minus signs when you compute the derivatives of $(1-t)^{-1/2}$.)

You can make this look nicer if you multiply top and bottom in the n^{th} term with 2^n:

$$T_\infty \frac{1}{\sqrt{1-t}} = 1 + \frac{1}{2}t + \frac{1\cdot 3}{2\cdot 4}t^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^3 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}t^n + \cdots$$

(IV9.27)

$$T_\infty \frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^6 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}t^{2n} + \cdots$$

(IV9.28)

$$T_\infty \arcsin t = t + \frac{1}{2}\frac{t^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{t^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{t^7}{7} + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\frac{t^{2n+1}}{2n+1} + \cdots$$

(IV9.29) $T_4[e^{-t}\cos t] = 1 - t + \frac{1}{3}t^3 - \frac{1}{6}t^4$.

(IV9.30) $T_4[e^{-t}\sin 2t] = t - t^2 + \dfrac{1}{3}t^3 + o(t^4)$ (the t^4 terms cancel).

(IV9.31) PFD of $1/(2-t-t^2) = \frac{1}{(2+t)(1-t)} = \frac{-\frac{1}{3}}{2+t} + \frac{\frac{1}{3}}{1-t}$. Use the geometric series.

(IV9.32) $\sqrt[3]{1+2t+t^2} = \sqrt[3]{(1+t)^2} = (1+t)^{2/3}$. This is very similar to problem **??**. The answer follows from Newton's binomial formula.

(IV11.1b) The Taylor series is

$$\sin(t) = t - t^3/6 + \cdots$$

and the order one and two Taylor polynomial is the same $p(t) = t$. For any t there is a ζ between 0 and t with

$$\sin(t) - p(t) = \frac{f^{(3)}(\zeta)}{3!} t^3$$

When $f(t) = \sin(t)$, $|f^{(n)}(\zeta)| \leq 1$ for any n and ζ. Consequently,

$$|\sin(t) - p(t)| \leq \frac{t^3}{3!}$$

for nonnegative t. Hence

$$|\int_0^{\frac{1}{2}} \sin(x^2)\ dx - \int_0^{\frac{1}{2}} p(x^2)\ dx| \leq \int_0^{\frac{1}{2}} |\sin(x^2) - p(x^2)|\ dx \leq \int_0^{\frac{1}{2}} \frac{x^6}{3!}\ dx = \frac{(\frac{1}{2})^7}{3!\ 7} = \epsilon$$

Since $\int_0^{\frac{1}{2}} p(x^2)\ dx = \frac{(\frac{1}{2})^3}{3} = A$ (the approximate value) we have that

$$A - \epsilon \leq \int_0^{\frac{1}{2}} \sin(x^2)\ dx \leq A + \epsilon$$

(IV11.2c) (b) $\frac{43}{30}$ (c) $\frac{3}{6!\cdot 13}$

(V3.1) 1/2 ($\lim_{n\to\infty} \frac{n}{2n-3} = \lim_{n\to\infty} \frac{n}{2n-3}\frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{2-\frac{3}{n}} = \frac{1}{2-0} = \frac{1}{2}$)

(V3.2) Does not exist (or "$+\infty$")

(V3.3) 1/2 ($\lim_{n\to\infty} \frac{n^2}{2n^2+n-3} = \lim_{n\to\infty} \frac{n^2}{2n^2+n-3}\frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}-\frac{3}{n^2}} = \frac{1}{2+0-0} = \frac{1}{2}$)

(V3.4) -1

(V3.5) 0 ($\lim_{n\to\infty} \frac{2^n+1}{1-3^n} = \lim_{n\to\infty} \frac{2^n+1}{1-3^n}\frac{\frac{1}{3^n}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{(\frac{2}{3})^n+\frac{1}{3^n}}{\frac{1}{3^n}-1} = \frac{0+0}{0-1} = 0$)

(V3.6) Does not exist (or "$-\infty$") because $e > 2$. ($\lim_{n\to\infty} \frac{e^n+1}{1-2^n} = \lim_{n\to\infty} \frac{e^n+1}{1-2^n}\frac{\frac{1}{2^n}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{(\frac{e}{2})^n+\frac{1}{2^n}}{\frac{1}{2^n}-1} = \frac{\infty+0}{0-1} = -\infty$) (Note: $\frac{\infty}{-1}$ is not an indeterminate form.)

(V3.7) 0. (For any polynomial n^p, $p > 0$, and any exponential a^n, $a > 1$, the exponential will go to ∞ much faster than the polynomial as n goes to ∞, regardless the size of p or a.) (Recall L'Hop Rule→the exponential does not change (in essence) while the polynomial keeps decreasing in power)

(V3.8) 0. (For any factorial $n!$, and any exponential a^n, $a > 1$, the factorial will go to ∞ much faster than the exponential as n goes to ∞, regardless the size of a.) (See final example in the section.)

(V3.9) 0 (write the limit as $\lim_{n\to\infty} \frac{n!+1}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} + \lim_{n\to\infty} \frac{1}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} + \lim_{n\to\infty} \frac{1}{(n+1)!}$).

(V3.10) 0 (Let $a_n = \frac{(n!)^2}{(2n)!}$. Then $0 \le a_{n+1} = \frac{((n+1)!)^2}{(2n+2)!} = \frac{(n!)^2}{(2n)!}\frac{(n+1)(n+1)}{(2n+1)(2n+2)} = a_n \frac{1}{2}\frac{n+1}{2n+1} < \frac{a_n}{2}$. (Every increase in n cuts a_n at least in half.) Then $\lim_{n\to\infty} a_n = 0$.

(V3.11) Use the explicit formula (**??**) from Example **??**. The answer is the Golden Ratio ϕ.

(V6.1) The remainder term $R_n(x)$ is equal to $\frac{f^{(n)}(\zeta_n)}{n!}x^n$ for some ζ_n. For either the cosine or sine and any n and ζ we have $|f^{(n)}(\zeta)| \le 1$. So $|R_n(x)| \le \frac{|x|^n}{n!}$. But we know $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ and hence $\lim_{n\to\infty} R_n(x) = 0$.

(V6.2) The k^{th} derivative of $g(x) = \sin(2x)$ is $g^{(k)}(x) = \pm 2^k \text{soc}(2x)$. Here $\text{soc}(\theta)$ is either $\sin\theta$ or $\cos\theta$, depending on k. Therefore k^{th} remainder term is bounded by

$$|R_k[\sin 2x]| \le \frac{|g^{(k+1)}(c)|}{(k+1)!}|x|^{k+1} = \frac{2^{k+1}|x|^{k+1}}{(k+1)!}|\text{soc}(2x)| \le \frac{|2x|^{k+1}}{(k+1)!}.$$

Since $\lim_{k\to\infty} \frac{|2x|^{k+1}}{(k+1)!} = 0$ we can use the Sandwich Theorem and conclude that $\lim_{k\to\infty} R_k[g(x)] = 0$, so the Taylor series of g converges for every x.

(V6.6) Read the example in §**??**.

(V6.7) We know that the Taylor's series of $f(x) = \dfrac{1}{1-x}$ is

$T^0_\infty(f(x)) = 1 + x + x^2 + x^3 \cdots = \sum_{n=0}^{\infty} x^n = \lim_{N\to\infty} \sum_{n=0}^{N} x^n$

Using the formula $\sum_{n=0}^{N} ar^n = \dfrac{a(1-r^{n+1})}{1-r}$.

$\sum_{n=0}^{N} x^n = \dfrac{1.(1-x^{n+1})}{1-x} = \dfrac{(1-x^{n+1})}{1-x}$

For **x=1**, $f(x)$ is undefined.

For $\mathbf{x} \le \mathbf{-1}$, $\sum_{n=0}^{N} x^n = \dfrac{(1-(x)^{n+1})}{2}$ oscillates about $1/2$ as $N \to \infty$. So the limit does not exist, ie. the Taylor series does not converge.

For $\mathbf{-1} < \mathbf{x} < \mathbf{1}$,

$$\lim_{N\to\infty} \sum_{n=0}^{N} x^n = \lim_{N\to\infty} \frac{(1-(x)^{n+1})}{2} = \frac{1}{2}$$

ie. the Taylor series converges.

(V6.8) Let $x^2 = u$. Then by the previous question the Taylor series converges when $-1 < u < 1$, ie. $-1 < x^2 < 1$, ie. when $-1 < x < 1$.

(V6.9) $-1 < x < 1$.

(V6.10) $-\frac{3}{2} < x < \frac{3}{2}$. Write $f(x)$ as $f(x) = \frac{1}{3}\frac{1}{1-(-\frac{2}{3}x)}$ and use the Geometric Series.

(V6.11) $|x| < 2/5$

(VI9.3e) (a) 3 (b) $\begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$ (c) 36 (d) $\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$ (e) $\begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix}$

(VI9.6c) (a) Since $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 1+x \\ 2+x \end{pmatrix}$ the number x would have to satisfy both $1 + x = 2$ and $2 + x = 1$. That's impossible, so there is no such x.

(b) No drawing, but $\vec{\boldsymbol{p}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the parametric representation of a straight line through the points $(1, 2)$ (when $x = 0$) and $(2, 3)$ (when $x = 1$).

(c) x and y must satisfy $\begin{pmatrix} x+y \\ 2x+y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Solve $x+y = 2$, $2x+y = 1$ to get $x = -1$, $y = 3$.

(VI9.7) Every vector is a position vector. To see of which point it is the position vector translate it so its initial point is the origin.

Here $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$, so $\overrightarrow{AB}$ is the position vector of the point $(-3, 3)$.

(VI9.8) One always labels the vertices of a parallelogram counterclockwise (see §**??**).

$ABCD$ is a parallelogram if $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$. $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. So $\overrightarrow{AB} + \overrightarrow{AD} \neq \overrightarrow{AC}$, and $ABCD$ is not a parallelogram.

(VI9.9a) As in the previous problem, we want $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$. If D is the point (d_1, d_2, d_3) then $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} d_1 \\ d_2 - 2 \\ d_3 - 1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$, so that $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$ will hold if $d_1 = 4$, $d_2 = 0$ and $d_3 = 3$.

(VI9.9b) Now we want $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$, so $d_1 = 4$, $d_2 = 2$, $d_3 = 5$.

(VI10.3b) (a) $\vec{\boldsymbol{x}} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-t \\ t \\ 1+t \end{pmatrix}$.

(b) Intersection with xy plane when $z = 0$, i.e. when $t = -1$, at $(4, -1, 0)$. Intersection with xz plane when $y = 0$, when $t = 0$, at $(3, 0, 1)$ (i.e. at A). Intersection with yz plane when $x = 0$, when $t = 3$, at $(0, 3, 4)$.

(VI10.4b) (a) $\vec{L}[t] = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$

(VI10.5c) (a) $\vec{p} = (\vec{b} + \vec{c})/2$, $\vec{q} = (\vec{a} + \vec{c})/2$, $\vec{r} = (\vec{a} + \vec{b})/2$.

(b) $\vec{m} = \vec{a} + \frac{2}{3}(\vec{p} - \vec{a})$ (See Figure **??**, with AX twice as long as XB). Simplify to get $\vec{m} = \frac{1}{3}\vec{a} + \frac{1}{3}\vec{b} + \frac{1}{3}\vec{c}$.

(c) Hint : find the point N on the line segment BQ which is twice as far from B as it is from Q. If you compute this carefully you will find that $M = N$.

(VI11.1) To decompose $\vec{b}$ set $\vec{b} = \vec{b}_\perp + \vec{b}_{/\!/}$, with $\vec{b}_{/\!/} = t\vec{a}$ for some number t. Take the dot product with $\vec{a}$ on both sides and you get $\vec{a}\ \vec{b} = t\|\vec{a}\|^2$, whence $3 = 14t$ and $t = \frac{3}{14}$. Therefore

$$\vec{b}_{/\!/} = \frac{3}{14}\vec{a}, \qquad \vec{b}_\perp = \vec{b} - \frac{3}{14}\vec{a}.$$

To find $\vec{b}_{/\!/}$ and $\vec{b} - \perp$ you now substitute the given values for $\vec{a}$ and $\vec{b}$.

The same procedure leads to $\vec{a}_\perp$ and $\vec{a}_{/\!/}$: $\vec{a}_{/\!/} = \frac{3}{2}\vec{b}$, $\vec{a}_\perp = \vec{a} - \frac{3}{2}\vec{b}$.

(VI11.2) This problem is of the same type as the previous one, namely we have to decompose one vector as the sum of a vector perpendicular and a vector parallel to the hill's surface. The only difference is that we are not given the normal to the hill so we have to find it ourselves. The equation of the hill is $12x_1 + 5x_2 = 130$ so the vector $\vec{n} = \binom{12}{5}$ is a normal.

The problem now asks us to write $\vec{f}_{\text{grav}} = \vec{f}_\perp + \vec{f}_{/\!/}$, where $\vec{f}_\perp = t\vec{n}$ is perpendicular to the surface of the hill, and $\vec{f}_{/\!/}$ id parallel to the surface.

Take the dot product with $\vec{n}$, and you find $t\|\vec{n}\|^2 = \vec{n}\ \vec{f}_{\text{grav}} \implies 169t = -5mg \implies t = -\frac{5}{169}mg$. Therefore

$$\vec{f}_\perp = -\frac{5}{169}mg\begin{pmatrix}12\\5\end{pmatrix} = \begin{pmatrix}-\frac{60}{169}mg\\-\frac{25}{169}mg\end{pmatrix}, \qquad \vec{f}_{/\!/} = \vec{f}_{\text{grav}} - \vec{f}_\perp = \begin{pmatrix}-\frac{60}{169}mg\\\frac{144}{169}mg\end{pmatrix},$$

(VI12.1c) $\|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 - 2\vec{a}\ \vec{b} + \|\vec{b}\|^2$; $\|2\vec{a} - \vec{b}\|^2 = 4\|\vec{a}\|^2 - 4\vec{a}\ \vec{b} + \|\vec{b}\|^2$; $\|\vec{a} + \vec{b}\| = \sqrt{54}$, $\|\vec{a} - \vec{b}\| = \sqrt{62}$ and $\|2\vec{a} - \vec{b}\| = \sqrt{130}$.

(VI12.3) Compute $\overrightarrow{AB} = -\overrightarrow{BA} = \begin{pmatrix}1\\1\end{pmatrix}$, $\overrightarrow{BC} = -\overrightarrow{CB} = \begin{pmatrix}-2\\2\end{pmatrix}$, $\overrightarrow{AC} = -\overrightarrow{CA} = \begin{pmatrix}-1\\3\end{pmatrix}$. Hence $\|\overrightarrow{AB}\| = \sqrt{2}$, $\|\overrightarrow{BC}\| = \sqrt{8} = 2\sqrt{2}$, $\|\overrightarrow{AC}\| = \sqrt{10}$.

And also $\overrightarrow{AB}\ \overrightarrow{AC} = 2 \implies \cos\angle A = \frac{\overrightarrow{AB}\ \overrightarrow{AC}}{\|\overrightarrow{AB}\|\ \|\overrightarrow{AC}\|} = \frac{2}{\sqrt{20}} = \frac{1}{\sqrt{5}}$.

A similar calculation gives $\cos\angle B = 0$ so we have a right triangle; and $\cos\angle C = \frac{2}{\sqrt{5}}$.

(VI12.4) $\overrightarrow{AB} = \binom{2}{1}$, $\overrightarrow{AC} = \binom{t-1}{2-t}$, $\overrightarrow{BC} = \binom{t-3}{1-t}$.

If the right angle is at A then $\overrightarrow{AB}\ \overrightarrow{AC} = 0$, so that we must solve $2(t-1) + (2-t) = 0$. Solution: $t = 0$, and $C = (0, 3)$.

If the right angle is at B then $\overrightarrow{AB}\ \overrightarrow{BC} = 0$, so that we must solve $2(t-3) + (1-t) = 0$. Solution: $t = 5$, and $C = (5, -2)$.

If the right angle is at C then $\overrightarrow{AC}\ \overrightarrow{BC} = 0$, so that we must solve $(t-1)(t-3) + (2-t)(1-t) = 0$. Note that this case is different in that we get a quadratic equation, and in that there are two solutions, $t = 1$, $t = \frac{5}{2}$.

This is a complete solution of the problem, but it turns out that there is a nice picture of the solution, and that the four different points C we find are connected with the circle whose diameter is the line segment AB:

(VI12.5a) ℓ has defining equation $-\frac{1}{2}x + y = 1$ which is of the form $\vec{n}\ \vec{x}$ =constant if you choose $\vec{n} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$.

(VI12.5b) The distance to the point D with position vector $\vec{d}$ from the line ℓ is $\frac{\vec{n}\ (\vec{d} - \vec{a})}{\|\vec{n}\|}$ where $\vec{a}$ is the position vector of any point on the line. In our case $\vec{d} = \vec{0}$ and the point $A(0,1)$, $\vec{a} = \overrightarrow{OA} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, is on the line. So the distance to the origin from the line is $\dfrac{-\vec{n}\ \vec{a}}{\|\vec{n}\|} = \dfrac{1}{\sqrt{(1/2)^2 + 1^2}} = 2/\sqrt{5}$.

(VI12.5c) $3x + y = 2$, normal vector is $\vec{m} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

(VI12.5d) Angle between ℓ and m is the angle θ between their normals, whose cosine is $\cos\theta = \frac{\vec{n}\ \vec{m}}{\|\vec{n}\|\ \|\vec{m}\|} = \frac{-1/2}{\sqrt{5/4}\sqrt{10}} = -\frac{1}{\sqrt{50}} = -\frac{1}{10}\sqrt{2}$.

(VI13.3a) $\vec{0}$ (the cross product of any vector with itself is the zero vector).

(VI13.3c) $(\vec{a} + \vec{b})\times(\vec{a} - \vec{b}) = \vec{a}\times\vec{a} + \vec{b}\times\vec{a} - \vec{a}\times\vec{b} - \vec{b}\times\vec{b} = -2\vec{a}\times\vec{b}$.

(VI13.4) Not true. For instance, the vector $\vec{c}$ could be $\vec{c} = \vec{a} + \vec{b}$, and $\vec{a}\times\vec{b}$ would be the same as $\vec{c}\times\vec{b}$.

(VI13.5a) A possible normal vector is $\vec{n} = \overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix} -4 \\ 4 \\ -4 \end{pmatrix}$. Any (non zero) multiple of this vector is also a valid normal. The nicest would be $\frac{1}{4}\vec{n} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$.

(VI13.5b) $\vec{n}\ (\vec{x} - \vec{a}) = 0$, or $\vec{n}\ \vec{x} = \vec{n}\ \vec{a}$. Using $\vec{n}$ and $\vec{a}$ from the first part we get $-4x_1 + 4x_2 - 4x_3 = -8$. Here you could replace $\vec{a}$ by either $\vec{b}$ or $\vec{c}$. (Make sure you understand why; if you don't think about it, then ask someone).

(VI13.5c) Distance from D to $\mathcal{P}$ is $\frac{\vec{n}\ (\vec{d} - \vec{a})}{\|\vec{n}\|} = 4/\sqrt{3} = \frac{4}{3}\sqrt{3}$. There are many valid choices of normal $\vec{n}$ in part (i) of this problem, but they all give the same answer here.

Distance from O to $\mathcal{P}$ is $\frac{\vec{n}\ (\vec{0} - \vec{a})}{\|\vec{n}\|} = \frac{2}{3}\sqrt{3}$.

(VI13.5d) Since $\vec{n}\ (\vec{0} - \vec{a})$ and $\vec{n}\ (\vec{d} - \vec{a})$ have the same sign the point D and the origin lie on the same side of the plane $\mathcal{P}$.

(VI13.5e) The area of the triangle is $\frac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\| = 2\sqrt{3}$.

(VI13.5f) Intersection with x axis is A, the intersection with y-axis occurs at $(0, -2, 0)$ and the intersection with the z-axis is B.

(VI13.6a) Since $\vec{\boldsymbol{n}} = \overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$ the plane through A, B, C has defining equation $-3x + y + z = 3$. The coordinates $(2, 1, 3)$ of D do not satisfy this equation, so D is not on the plane ABC.

(VI13.6b) If E is on the plane through A, B, C then the coordinates of E satisfy the defining equation of this plane, so that $-3\cdot 1 + 1\cdot 1 + 1\cdot\alpha = 3$. This implies $\alpha = 5$.

(VI13.7a) If $ABCD$ is a parallelogram then the vertices of the parallelogram are labeled A, B, C, D as you go around the parallelogram in a counterclockwise fashion. See the figure in §43.2.

Then $\overrightarrow{AB}+\overrightarrow{AD} = \overrightarrow{AC}$. Starting from this equation there are now two ways to solve this problem.

(first solution) If D is the point (d_1, d_2, d_3) then $\overrightarrow{AD} = \begin{pmatrix} d_1-1 \\ d_2+1 \\ d_3-1 \end{pmatrix}$, while $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$. Hence $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$ implies $\begin{pmatrix} d_1 \\ d_2+2 \\ d_3-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$, and thus $d_1 = 0$, $d_2 = 1$ and $d_3 = 0$.

(second solution) Let $\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}, \vec{\boldsymbol{d}}$ be the position vectors of A, B, C, D. Then $\overrightarrow{AB} = \vec{\boldsymbol{b}} - \vec{\boldsymbol{a}}$, etc. and $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$ is equivalent to $\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}} + \vec{\boldsymbol{d}} - \vec{\boldsymbol{a}} = \vec{\boldsymbol{c}} - \vec{\boldsymbol{a}}$. Since we know $\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}$ we can solve for $\vec{\boldsymbol{d}}$ and we get $\vec{\boldsymbol{d}} = \vec{\boldsymbol{c}} - \vec{\boldsymbol{b}} + \vec{\boldsymbol{a}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(VI13.7b) The area of the parallelogram $ABCD$ is $\|\overrightarrow{AB}\times\overrightarrow{AD}\| = \left\|\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}\right\| = \sqrt{11}$.

(VI13.7c) In the previous part we computed $\overrightarrow{AB}\times\overrightarrow{AD} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$, so this is a normal to the plane containing A, B, D. The defining equation for that plane is $-x + y + 3z = 1$. Since $ABCD$ is a parallelogram any plane containing ABD automatically contains C.

(VI13.7d) $(-1, 0, 0)$, $(0, 1, 0)$, $(0, 0, \frac{1}{3})$.

(VI13.8a) Here is the picture of the parallelepiped (which you can also find on page 103): Knowing the points A, B, D we get $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. Also, since $\substack{EFGH \\ ABCD}$ is a parallelepiped, we know that all its faces are parallelogram, and thus $\overrightarrow{EF} = \overrightarrow{AB}$, etc. Hence: we find these coordinates for the points A, B, ...

$A(1, 0, 0)$, (given); $B(0, 2, 0)$, (given); $C(-2, 2, 1)$, since $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$; $D(-1, 0, 1)$, (given); $E(0, 0, 2)$, (given)
$F(-1, 2, 2)$, since we know E and $\overrightarrow{EF} = \overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

$G(-3,2,3)$, since we know F and $\overrightarrow{FG} = \overrightarrow{EH} = \overrightarrow{AD} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$
$H(-2,0,3)$, since we know E and $\overrightarrow{EH} = \overrightarrow{AD} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

(VI13.8b) The area of $ABCD$ is $\|\overrightarrow{AB}\times\overrightarrow{AD}\| = \sqrt{21}$.

(VI13.8c) The volume of $\mathfrak{P}$ is the product of its height and the area of its base, which we compute in the previous and next problems. So height$= \frac{\text{volume}}{\text{area base}} = \frac{6}{\sqrt{21}} = \frac{2}{7}\sqrt{21}$.

(VI13.8d) The volume of the parallelepiped is $\overrightarrow{AE}\ (\overrightarrow{AB}\times\overrightarrow{AD}) = 6$.