Solutions Exam 3

1. (a) First we divide through by x^2+5, so

$$y' - \frac{x}{x^2+5}\ y = \frac{x}{x^2+5}.$$

The integrating factor is $e^{\int -\frac{x\ dx}{x^2+5}} = e^{-\frac{1}{2}\log(x^2+5)} = (x^2+5)^{-1/2}$, where we substituted $u = x^2+5$.

$$\begin{aligned} y' - x(x^2+5)^{-1}y &= x(x^2+5)^{-1} \\ (x^2+5)^{-1/2}y' - x(x^2+5)^{-3/2}y &= x(x^2+5)^{-3/2} \\ [(x^2+5)^{-1/2}y]' &= x(x^2+5)^{-3/2} \\ (x^2+5)^{-1/2}y &= \int x(x^2+5)^{-3/2}\ dx = -(x^2+5)^{-1/2} + C \\ y &= -1 + C(x^2+5)^{1/2}. \end{aligned}$$

Using the initial condition, $0 = -1 + 3C$, so $C = 1/3$, so

$$y = \frac{1}{3}\sqrt{x^2+5} - 1.$$

(b)

$$\begin{aligned} (x^2+5)\frac{dy}{dx} - xy &= x \\ (x^2+5)\frac{dy}{dx} &= xy + x = x(y+1) \\ \frac{dy}{y+1} &= \frac{x\ dx}{x^2+5} \\ \int \frac{dy}{y+1} &= \int \frac{x\ dx}{x^2+5} \\ \log(y+1) &= \frac{1}{2}\log(x^2+5) + C. \end{aligned}$$

Using the initial condition, $0 = \frac{1}{2}\log 9 + C$, so $C = -\frac{1}{2}\log 9 = \log(9^{-1/2}) = \log\frac{1}{3}$.

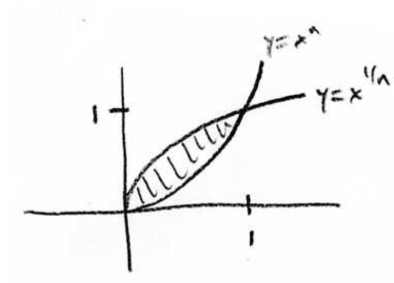
$$\begin{aligned} \log(y+1) &= \frac{1}{2}\log(x^2+5) + \log\frac{1}{3} = \log\frac{1}{3}(x^2+5)^{1/2} \\ y+1 &= \frac{1}{3}(x^2+5)^{1/2} \\ y &= \frac{1}{3}\sqrt{x^2+5} - 1 \end{aligned}$$

which agrees with our previous answer.

(c)

$$(x^2+5)y' - xy = (x^2+5)[x(x^2+5)^{-1/2}] - x[(x^2+5)^{1/2} - 1] = x(x^2+5)^{1/2} - x(x^2+5)^{1/2} + x = x.$$

2. (a) The picture is essentially the same for any n. The curves intersect when $x^{1/n} = x^n$, so $x = x^{n^2}$, so either $x = 0$ or we can divide by x, in which case $1 = x^{n^2-1}$, so $x = 1^{1/(n^2-1)} = 1$. Maybe there is also a negative root, but we can ignore it since we're in the first quadrant. If x is between 0 and 1, x^n is below $x^{1/n}$. For larger n, the lower curve is lower and the upper curve is higher.



(b) The region is bounded by $y = x^n$ and $y = x^{1/n}$. If we solve for x, these become $x = y^{1/n}$ and $x = y^n$. Switching x and y gives us back the original equations, so if we switch the x- and y-axes we'll get the same volume.

(c) We rotate around the x-axis. The upper curve gives disks of volume $\pi x^{2/n}\ dx$, and the lower curve gives disks of volume $\pi x^{2n}\ dx$. Thus the volume of the solid is

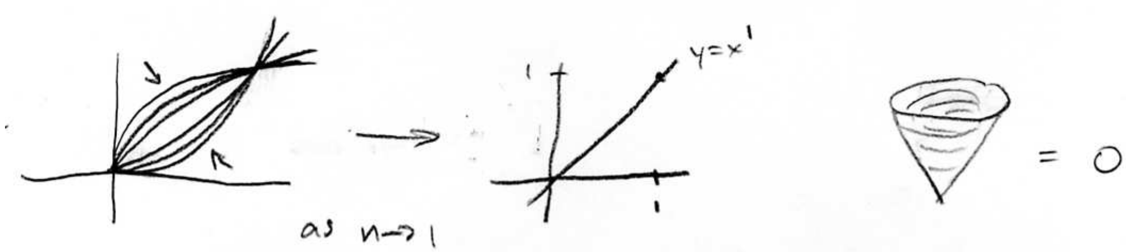
$$\int_0^1 \pi x^{2/n}\ dx - \int_0^1 \pi x^{2n}\ dx = \pi \frac{x^{(2/n)+1}}{(2/n)+1}\bigg|_0^1 - \pi \frac{x^{2n+1}}{2n+1}\bigg|_0^1$$
$$= \pi \frac{1}{(2/n)+1} - \pi \frac{1}{2n+1} = \pi \frac{n}{n+2} - \pi \frac{1}{2n+1} = \pi \frac{2n^2-2}{(n+2)(2n+1)}.$$

(d) We rotate instead around the y-axis. A typical shell has radius x and height $x^{1/n} - x^n$, so the volume of the solid is

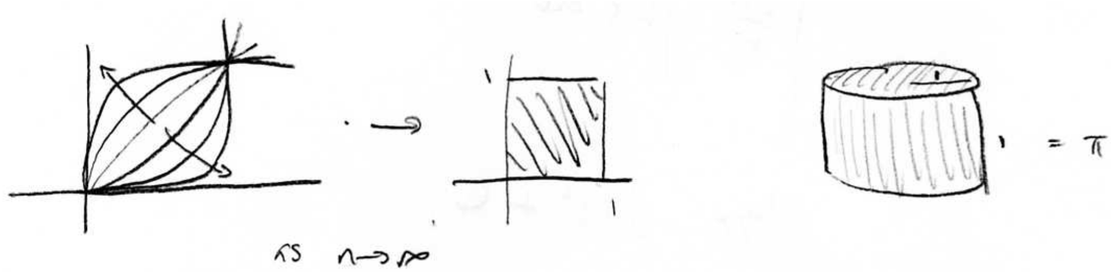
$$\int_0^1 2\pi x(x^{1/n} - x^n)\ dx = 2\pi \int_0^1 (x^{(1/n)+1} - x^{n+1})\ dx = 2\pi \left[\frac{x^{(1/n)+2}}{(1/n)+2} - \frac{x^{n+2}}{n+2}\right]_0^1$$
$$= 2\pi \left(\frac{1}{(1/n)+2} - \frac{1}{n+2}\right) = 2\pi \left(\frac{n}{2n+1} - \frac{1}{n+2}\right) = 2\pi \frac{n^2-1}{(n+2)(2n+1)}$$

which agrees with our previous answer.

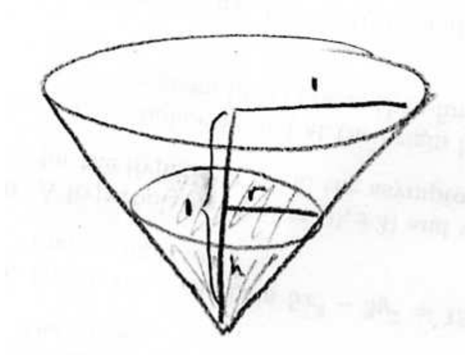
(e) As $n \to 1$, the volume tends to $0/2 = 0$, which makes sense because the region between $y = x$ and $y = x$ has no area:



As $n \to \infty$, the volume tends to π, which makes sense because the region becomes a square:



3. (a)



(b) $\dfrac{dV}{dt} = k\sqrt{h}$.

(c) $\dfrac{r}{h} = \dfrac{1}{1}$, so $r = h$.

(d) $V = \dfrac{1}{3}\pi h^3$.

(e) $\dfrac{dV}{dt} = \pi h^2 \dfrac{dh}{dt}$.

(f) $\pi h^2 \dfrac{dh}{dt} = k\sqrt{h}$.

(g)

$$\begin{aligned}
\pi h^2 \frac{dh}{dt} &= k h^{1/2} \\
h^{3/2}\ dh &= \frac{k}{\pi}\ dt \\
\int h^{3/2}\ dh &= \int \frac{k}{\pi}\ dt \\
\frac{2}{5} h^{5/2} &= \frac{k}{\pi} t + C.
\end{aligned}$$

When $t = 0$, $h = 1$, so $C = 2/5$. When $t = 1$, $h = 0$, so $0 = \dfrac{k}{\pi} + \dfrac{2}{5}$, so $\dfrac{k}{\pi} = -\dfrac{2}{5}$. Thus

$$\begin{aligned}
\frac{2}{5} h^{5/2} &= -\frac{2}{5} t + \frac{2}{5} \\
h^{5/2} &= -t + 1 = 1 - t \\
h &= (1-t)^{2/5}.
\end{aligned}$$

(h) The volume of the tank is $\pi/3$, so when the tank is half full $\frac{\pi}{6} = \frac{\pi}{3} h^3$, so $h^3 = \frac{1}{2}$, so $h = (\frac{1}{2})^{1/3}$.

(i)

$$\begin{aligned}
(\tfrac{1}{2})^{1/3} &= (1-t)^{2/5} \\
(\tfrac{1}{2})^{5/6} &= 1 - t \\
t &= 1 - (\tfrac{1}{2})^{5/6} \approx .44 \text{ hours} \approx 26 \text{ minutes}.
\end{aligned}$$

The tank is half full about 26 minutes after it starts draining, which makes sense: the tank drains faster when the water is deeper, so the first half of the water should drain out faster than the second half.