Solutions to Problem Set 12

I. Problems to be graded on completion.

1. a. The average value is

$$\frac{1}{1-(-1)}\int_{-1}^{1} x^2\ dx = \frac{1}{2}\frac{x^3}{3}\bigg|_{-1}^{1} = \frac{1}{3}.$$

The function takes this value when $x = \pm\frac{1}{\sqrt{3}} \approx \pm .577$.

b. The average value is

$$\frac{1}{1-(-1)}\int_{-1}^{1} x^3\ dx = \frac{1}{2}\frac{x^4}{4}\bigg|_{-1}^{1} = 0.$$

The function takes this value when $x = 0$.

c. The average value is

$$\frac{1}{4-1}\int_{1}^{4} \frac{1}{x}\ dx = \frac{1}{3}\log x\bigg|_{1}^{4} = \frac{\log 4}{3} \approx .462.$$

The function takes this value when $x = \frac{3}{\log 4} \approx 2.16$.

d. The average value is

$$\frac{1}{\pi-0}\int_{0}^{\pi} \sin x\ dx = -\frac{1}{\pi}\cos x\bigg|_{0}^{\pi} = \frac{2}{\pi} \approx .637.$$

The function takes this value when $x = \arcsin(2/\pi) \approx .69$ or $\pi - \arcsin(2/\pi) \approx 2.45$.

e. The average value is

$$\frac{1}{9-1}\int_{1}^{9} x^{1/2}\ dx = \frac{1}{8}\frac{x^{3/2}}{3/2}\bigg|_{1}^{9} = \frac{13}{6} = 2 + \frac{1}{6}.$$

The function takes this value when $x = \frac{169}{36} = 4 + \frac{25}{36}$.

2. $6 = k \cdot \frac{1}{2}$, so $k = 12$.

$$\int_0^2 12x \; dx = 6x^2 \Big|_0^2 = 24 \text{ ft-lbs.}$$

4. Let k be the spring constant and l be the natural length. Then

$$.05 = \int_8^9 k(x-l) \; dx = \frac{1}{2}k(x-l)^2 \Big|_8^9 = \frac{1}{2}k[(9-l)^2 - (8-l)^2] = \frac{1}{2}k(17-2l)$$

so $\frac{.10}{k} = 17 - 2l$. Next,

$$.10 = \int_9^{10} k(x-l) \; dx = \frac{1}{2}k(x-l)^2 \Big|_9^{10} = \frac{1}{2}k(19-2l)$$

so $\frac{.20}{k} = 19 - 2l$. Thus $19 - 2l = 2(17-2x) = 34 - 4l$, so $2l = 15$, so $l = \frac{15}{2} = 7.5$. The spring constant is $k = \frac{.10}{17 - 2\frac{15}{2}} = .05$.

10. Let x be the distance from the top of the tank. For each x, the volume of a slice is $\frac{4}{3}x \cdot 10 \cdot dx = \frac{40}{3}x \; dx$, so the weight of a slice is $62.4 \cdot \frac{40}{3}x \; dx = 832x \; dx$, so the work done removing it from the tank is $x \cdot 832x \; dx = 832x^2 \; dx$. The work done in pumping out the whole tank is

$$\int_0^3 832x^2 \; dx = \frac{832}{3}x^3 \Big|_0^3 = 7488 \text{ ft-lbs.}$$

12. Again let x be the distance from the top of the tank. For each x, the volume of a slice is $\sqrt{9-x^2} \cdot 10 \cdot dx = 10\sqrt{9-x^2} \; dx$, so the weight of a slice is $62.4 \cdot 10\sqrt{9-x^2} \; dx = 624\sqrt{9-x^2} \; dx$, so the work done removing it from the tank is $x \cdot 624\sqrt{9-x^2} \; dx = 624x\sqrt{9-x^2} \; dx$. The work done in pumping out the whole tank is

$$\int_0^3 624x\sqrt{9-x^2} \; dx = \int_9^0 -312\sqrt{u} \; du = 312\int_0^9 u^{1/2} \; du = 312\frac{u^{3/2}}{3/2}\Big|_0^9 = 5616 \text{ ft-lbs.}$$

where we let $u = 9 - x^2$, so $du = -2x \; dx$.

19. Let x be the distance from the bucket at the end of the cable to the axle at the top of the shaft. At each distance x, we are hauling 200 lbs of bucket and $2x$ lbs of chain, for a total of $(200 + 2x) \; dx$ ft-lbs of work. The total work is

$$\int_0^{500} (200 + 2x) \; dx = \Big[200x + x^2\Big]_0^{500} = 350,000 \text{ ft-lbs.}$$

21. At each height x, we are doing $\dfrac{k}{x^2} \; dx$ mi-lbs of work, so the total work is

$$\int_{4000}^{4200} \frac{k}{x^2} \; dx = -kx^{-1}\Big|_{4000}^{4200} = \frac{k}{84000}$$

If $5000 = k/(4000)^2$ then $k = 8 \times 10^{10}$, so the total work is $\frac{2 \times 10^{10}}{21} \approx 952,381$ mi-lbs.

II. Problems to be graded on correctness.

1. The mass of the rod is

$$\int_0^6 12(x+1)^{-1/2}\,dx = \int_1^7 12u^{-1/2}\,du = 12\frac{u^{1/2}}{1/2}\Big|_1^7 = 24(\sqrt{7}-1) \approx 39.5.$$

where we let $u = x+1$, so $du = dx$. The center of mass is

$$\frac{1}{24(\sqrt{7}-1)}\int_0^6 12x(x+1)^{-1/2}\,dx = \frac{1}{2(\sqrt{7}-1)}\int_1^7 (u-1)u^{-1/2}\,du = \frac{1}{2(\sqrt{7}-1)}\int_1^7 (u^{1/2}-u^{-1/2})\,du$$

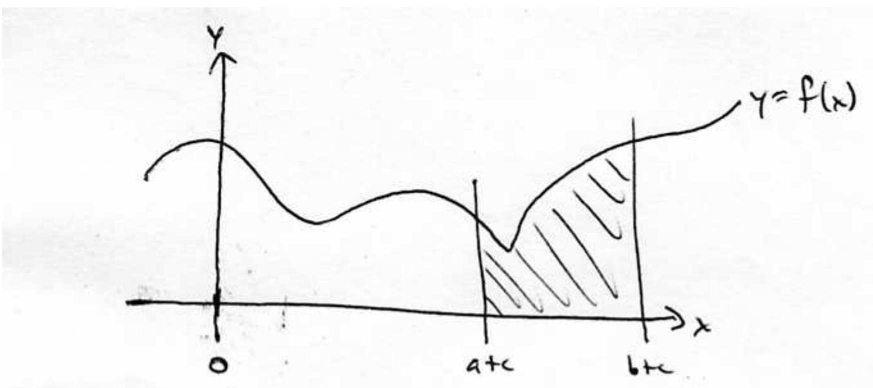
$$= \frac{1}{2(\sqrt{7}-1)}\left[\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right]_1^7 = \frac{4\sqrt{7}+2}{3(\sqrt{7}-1)} = \frac{\sqrt{7}+5}{3} \approx 2.55.$$

20. Let x be the monkey's height above his original position. At each height x he is lifting his own weight (10 lbs) plus $\frac{1}{2}\cdot\frac{1}{2}x$ lbs of chain, so he is doing $(10+\frac{1}{4}x)\,dx$ ft-lbs of work. The total work is

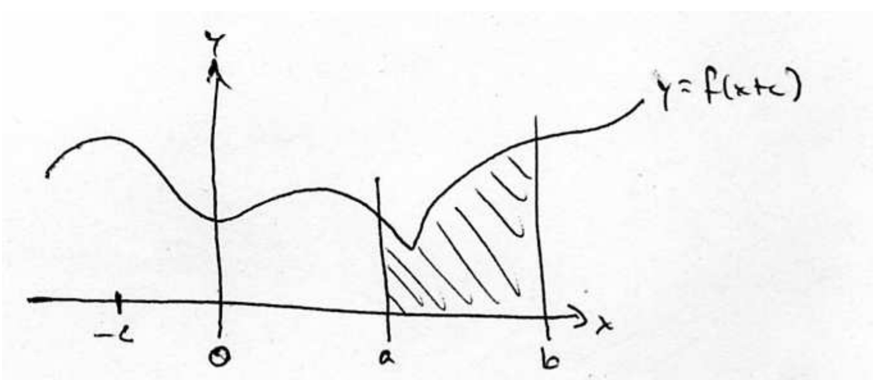
$$\int_0^{20}\left(10+\frac{1}{4}x\right)dx = \left[10x+\frac{1}{8}x^2\right]_0^{20} = 250 \text{ ft-lbs.}$$

2.

a, c.



b, d.



e. Let $u = x+c$, so $du = dx$. When $x = a$, $u = a+c$. When $x = b$, $u = b+c$.

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(u)\,du.$$

f. It moves the graph c units to the right, but it also moves the bounds c units to the right, so it's the same area.